[illegible] [illegible]

[illegible]
200

APPLIED MECHANICS

BY

CHARLES E. FULLER, S. B.

AND

WILLIAM A. JOHNSTON, S. B.

PROFESSORS OF THEORETICAL AND APPLIED MECHANICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

VOL. I

STATICS AND KINETICS

FIRST EDITION

TOTAL ISSUE FIVE THOUSAND

NEW YORK
JOHN WILEY & SONS, INC.
LONDON: CHAPMAN & HALL, LIMITED
1915

Stanhope Press
F. H. GILSON COMPANY
BOSTON, U.S.A.

PREFACE

THIS volume is the outgrowth of an extended experience in teaching the subject of Applied Mechanics at the Massachusetts Institute of Technology and has been written primarily for students who have had a preliminary training in the fundamentals of Mechanics, in a course in Physics, and also in a course in the Elements of Mechanicm.

The title "Applied Mechanics" has been adopted as best fitting the course in which the text of the present volume forms the first part. A thorough discussion of the underlying theory, however, in so far as it will help to elucidate the principles which are required in the solution of problems in Engineering, is attempted. For the sake of completeness, a brief statement of the fundamental conceptions of motion, mass and force, with which the student is expected to be familiar, is given in the Introductory Chapter and, for a similar reason, some subject matter has been included, notably in the last part of Chapters II, IV, and V, which may properly be taken up in a later course.

Some freedom has been exercised in the choice of the titles for the different chapters, allied parts of the work having been presented under five general headings. For example, in Chapter II, the methods of resolution and composition of forces are discussed as a basis of the conditions of equilibrium which apply to problems in pure Statics. Also, Chapter V on Kinetics, includes a presentation of certain of the elementary principles of Kinematics which are required in the discussion of the later theories belonging strictly to the subject of Kinetics. The principles of Work and Energy and the Laws of Friction and applications are also included in this chapter, as being closely allied to the subject of the Kinetics of Rigid Bodies.

A large number of problems, some simple, others more complex, illustrating the application of the principles discussed in the text, have been included and solutions have been given when it was

deemed necessary to indicate the methods of applying the principles. It is the experience of the authors, that the student's power to clearly grasp the subject comes, to a large extent, from his efforts to solve original problems.

Throughout the text, the methods of the Calculus have been employed when these appear to furnish the most direct method of analysis. In general, however, only the simpler operations in differentiation and integration are required.

The term vector has been used to apply to the geometrical line representing a force, moment, velocity, acceleration, etc.; it being assumed that the student is familiar with the simple operations of adding and subtracting vectors.

In the use of mathematical terms and expressions, we have endeavored to convey the meaning, in the simplest manner, and have not always employed the forms which a rigorous mathematical treatment of the subject might require. In the nomenclature, we have confined ourselves to the use of the terms employed by some of the older writers on Applied Mechanics, instead of adopting those usually employed in works on Theoretical Mechanics.

Finally, the authors wish to express their indebtedness to Prof. Lanza for the aid and many helpful suggestions it has been their good fortune to receive during their long association with him as students and in their work at the Massachusetts Institute of Technology.

C. E. F. and W. A. J.

August, 1913.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTORY

CHAPTER II

STATICS

§1. Preliminary

CHAPTER IV

MOMENT OF INERTIA

CHAPTER V

KINETICS

§1. KINEMATICS: KINETICS OF THE PARTICLE

§2. Work, Power and Energy

§3. Friction

§4. Kinetics of Rigid Bodies Having Plane Motion Only

APPLIED MECHANICS

CHAPTER I.

INTRODUCTORY.

1. Mechanics. — Mechanics, in the sense in which the term is ordinarily used, may be defined as the science which treats of the motion of bodies in space, and the effects of forces in modifying motion. It is a natural science, the foundations of which are a few laws or first principles, which are the same for all bodies, and which have been determined by observation and experiment. Upon these fundamental principles as axioms, by logical reasoning, largely by the application of the methods of mathematics, the whole superstructure of the various parts of the subject has been built.

In this work, only those parts of the subject which relate to the solution of certain problems in Engineering will be treated, and the mathematical deductions are carried out with the object of applying them to the solution of these problems. Owing to the fact that there are certain limitations and many approximations which must be made in the practical application of the results of these deductions, the treatment is briefer and less complete than would be the case if the subject were to be considered on its theoretical side alone.

2. Division of Mechanics. — The subject of Mechanics is subdivided into the subjects of *Kinematics* and *Dynamics*.

Kinematics is the science which treats of the motion only, without regard to the cause.

Dynamics is the science which treats of the action of forces in producing or preventing motion. It may be subdivided into two parts, *Statics* and *Kinetics*.

Statics is the science which treats of systems of forces which are in equilibrium.

Kinetics is the science which treats of the effects of forces in producing, or modifying, motion in bodies.

In some works on Mechanics, the term Dynamics has been restricted to the subject of Kinetics alone, Statics being treated as an independent subject in which the laws governing the equilibrium of systems of forces are deduced by the methods of mathematics, based on the fundamental conception of force as a "pressure" acting on a body.

3. Order of Treatment. — The commonly accepted order of treatment of these subjects is Kinematics, Statics, Kinetics. In this work the subject of Statics will be taken up first. A few propositions in Kinematics will be included under the subject of Kinetics. At times the broader term Dynamics may be used in the place of Kinetics in dealing with that part of the subject. A brief treatment of the methods of determining centers of gravity and moments of inertia will also be included.

4. Additional Division of Mechanics. — While the general underlying principles of Mechanics are the same when applied to solid, liquid, or gaseous bodies; special methods are developed in treating these different cases. Hence it is advantageous to treat separately: (a) the Mechanics of solid bodies which may be considered as rigid, (b) the Mechanics of non-rigid solid bodies, (c) the Mechanics of liquids, (d) the Mechanics of gases.

5. Fundamental Conceptions of Mechanics. — The fundamental conceptions of Mechanics are Matter, Space, Time, Motion and Force.

6. Motion. — Motion is change of position in space. All motion is relative. We have no means of determining the absolute motion of a body, but can determine its motion only in relation to that of some point, which for convenience we may consider as fixed.

In problems in Engineering the so-called fixed point may be a definite point which is at rest in relation to the Earth, or it may be a point on a body which is in motion relatively to the Earth.

A body may be considered at *rest* in relation to some fixed point when the directions and distances of all points in the body from that point remain constant.

7. Velocity. — Velocity may be defined as the rate of motion of one point with reference to another. If the rate is constant so that the space traversed is proportional to the time the velocity is said to be uniform. It may be expressed analytically by the formula,

$$v = \frac{ds}{dt},$$

which will apply for nonuniform as well as uniform velocity.

8. Acceleration. — Acceleration is the rate of change of velocity. If the rate is constant the acceleration is said to be uniform. It may be expressed analytically by the formula

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

9. Mass and Density. — The term *mass* is commonly used to denote the quantity of matter that a body contains.

For a homogeneous body the ratio of the mass to the volume, that is, the mass per unit of volume, is called its *density*. For a non-homogeneous body the density may be expressed by the formula $d = \frac{dM}{dV}$, where d is the limit of the ratio of a very small mass dM to its volume dV, as dV approaches zero.

10. Momentum. — The product of the mass of a body and its velocity is called its *momentum*.

11. Force. — *Force, in the sense in which the term is used in Mechanics, is the tendency to change the relative motion of two bodies between which that tendency exists.*

This conception of force is based on the fundamental laws of motion, first stated by Newton in the "Principia," which, quoted in translation, are as follows:

LAW I. — "Every body continues in its state of rest, or of uniform motion in a straight line, except in so far as it may be compelled by force to change that state."

LAW II. — "Change of motion is proportional to the force applied and takes place in the direction in which the force acts."

LAW III. — "To every action there is always an equal and contrary reaction; or, the mutual actions of two bodies are always equal and oppositely directed."

Law I is a statement of the principle of the *inertia of matter;* Law II is the foundation of *Kinetics;* and Law III is commonly called the law of *action and reaction.* As is the case with all

physical principles, these laws are known to be true only from observation and experiment. Since all our ideas of motion are relative, we can only recognize force as acting when at least two bodies are concerned in the transaction. We cannot say that force is an absolute quantity, such as a pressure acting on a body which is independent of all the surrounding bodies; for we can only recognize force from the effect produced. As we cannot measure absolute motion, but only the motion of one body relatively to another, we must regard the cause of the change of the motion of the one body in relation to the other, which we call force, as a relative quantity.

Two or more bodies may exert forces on a given body simultaneously, and their relations may be such that no change in motion will result; but each force will have the same tendency to produce motion as if it acted alone. *Such a body is said to be in equilibrium under the forces acting upon it.*

12. External Forces. — In the solution of any problem in Mechanics, we may confine our attention wholly to the motion, or condition, of one of the two bodies between which a force acts. The motion may be referred to the other body as fixed, or to some point outside of either. A force would then be spoken of as an *external force* acting on the body under consideration. This method of dealing with forces gives rise to the statements in common use, such as: a force acts on a given body; at a given point; in a given plane; etc. which, while not logically correct, are employed for the sake of simplicity.

When a number of external forces are exerted simultaneously on a body in such a manner that no change in motion occurs, *the system of forces is said to be in equilibrium*, or it may be referred to as *a balanced system of forces.*

13. Dynamical Measure of Force. — Newton states in the "Principia": "If any force generates any momentum, a double force will generate a double, a triple force will generate a triple momentum, whether simultaneously and suddenly, or gradually and successively impressed. And if the body was moving before: this momentum, if in the same direction as the motion, is added; if opposite, is subtracted; or if in an oblique direction, is annexed obliquely, and compounded with it, according to the direction and magnitude of the two."

This has been incorporated in the more modern statement of

the second law of motion, and its corollary which may be stated as follows:

LAW II. — *Forces are proportional to the momenta which would be generated by their constant and uniform action during a unit of time.*

COROLLARY. — *When any number of forces act simultaneously upon a body, then, whether the body be originally at rest or in motion, each force produces exactly the same effect in magnitude and direction as if it acted alone.*

The momentum produced in a unit of time may, therefore, be taken as the measure of the force producing it, and if F equals the force acting, M, the mass and a, its acceleration we may write

$$F = Ma,$$

provided the units for M and a are so chosen that F is unity when M and a are unity.

14. Measure of Mass. — Since $M = \frac{F}{a}$, the mass of a body may be measured by the ratio between any force acting upon it and the acceleration which the force produces.

If W is the force by which a mass M is drawn toward the Earth and g the acceleration produced by that force,

$$M = \frac{W}{g}.$$

The unit mass will, therefore, weigh g units of force, whatever system of units may be adopted.

15. Systems of Units. — Owing to the fact that the force exerted between the Earth and any given mass varies with the distance between their centers, two principal systems of measurement of force and mass have been used, called respectively the Absolute and Gravitation systems. In both systems the unit of time is the second; and either the metric or British measures of weight, mass and distance have been used.

Absolute System. — In the absolute system the unit of mass is assumed and the unit of weight determined from the equation $M = \frac{W}{g}$. In the *Centimeter-Gram-Second System,* the mass of the standard gram is chosen as the unit mass. Since $M = \frac{W}{g}$, the unit of force will be equal to $\frac{1}{g}$ part of the weight of the unit mass in

any locality. This unit is called the *dyne*, and will evidently be the same in all localities. As is indicated by the name given to the system, the centimeter is the unit of length. This is the particular form of absolute system now universally employed in scientific work.

Gravitation System. —.In the gravitation system the unit of weight is assumed and the unit of mass derived from the equation $M = \frac{W}{g}\cdot$ Where the British units are used, the weight of the standard pound is chosen as the unit of force. The unit mass will therefore weigh g pounds. Since the force exerted by the Earth on the pound weight varies with the locality; it is evident that the unit of force in this system is variable, unless the weight of the pound at some particular locality is taken as the unit, which has not been the case.

This system, however, is almost universally employed in engineering work, the variation in the force of gravity being so small as to be negligible. It is customary to use a constant value of 32.2 ft. per sec^{-2} for the value of g. A similar system in metric units is also in use.

16. Statical Measure of Force. — The units of measurement of balanced forces are evidently the same as the dynamical units. The pound and kilogram, of the *gravitation system*, are the units almost universally employed in engineering computations in Statics.

The direct measurement of a force is commonly accomplished by one of two methods. The first is based on the principles of leverage, and the second on the resistance of a spring to tension or compression. Lever balances and the common platform weighing scales are examples of the first method; while spring balances, steam engine indicators and various types of spring dynamometers are examples of the second method of measuring force.

17. Weight. — Since the weight of a body is the force exerted upon it by the Earth, the units of weight are the same as units of force.

18. Variation in the Gravitation Unit of Force. — The value of the acceleration due to gravity at different parts of the Earth is given by the following formula, in which g is the acceleration in feet per second, L the latitude and H the elevation above sea level in feet.

$$g = 32.0894\,(1 + 0.005243 \sin^2 L)(1 - 0.0000000957\,H).$$

The weight of the standard pound or kilogram will vary directly as g varies. The extreme variation, between the values at the highest point on the equator and the pole, would be somewhat less than two-thirds of one per cent. The maximum variation within the limits of the United States is less than one-third of one per cent. This variation is within the limits of accuracy of all ordinary engineering computations.

19. Characteristics of Force. — In the light of the preceding discussion it will be seen that a force is a vector quantity. It has *magnitude*, *direction* and *place of application.*

Action and Reaction. — According to Newton's Third Law of Motion (Art. 11), when one body exerts a force on another, the latter body exerts an equal force, opposite in direction, on the first. One of these forces may be called the *action* and the other the *reaction.*

20. Distributed and Concentrated Forces. — *A distributed force* is one which acts on a surface; such as the pressure of water on the side of a vessel, the pressure of a column on its foundation: or, which acts through a given volume; such as the attraction of the Earth on a weight.

The *place of application* of the force, in the first case, is the surface over which the pressure acts; and in the second, the volume through which the force of gravity acts.

All forces are really distributed forces. In many cases however it is convenient to consider a force, whose place of application is small, as though it were applied at a point. Such a force is called a *concentrated force.* No finite force can act at a single point, but, in the solution of a large number of problems, forces may be treated as though acting at single points. Distributed forces may also be treated as if made up of a very large number of small concentrated forces.

The line of action of a concentrated force is a line passing through its point of application in the direction in which the force acts.

Both distributed and concentrated forces may be divided into the two following groups: (a) *Forces acting between bodies which are in contact.* (b) *Forces acting between bodies which are at some distance apart.* Examples of force by contact are the pressure of the atmosphere on any object, the pressure of steam on a piston, the load acting on a column, etc. Examples of force acting at a distance are gravitational, electrical and magnetic forces.

21. Graphical Representation of Force. — Since a concentrated force is a vector quantity, it may be represented by a straight line, or vector, drawn parallel to its line of action. The length of the line represents, to some convenient scale, the magnitude of the force; the direction of the force being indicated by an arrow placed on the line. In some cases the vector representing the force is drawn to coincide with the line of action of the force; but in most cases it will be found more convenient, or even necessary, to draw it parallel to the line of action.

Unless otherwise stated, we shall always consider forces as concentrated and denote them by such letters as F_1, F_2, F_3, P_1, P_2, P_3, R_1, R_2, etc., or such other notation as may be suited to the problem.

22. Composition of Forces. — Parallelogram of Forces. — The corollary of Newton's second law of motion might be expressed as follows:

If a body have two or more velocities imparted to it simultaneously, it will move so as to preserve them all.

This principle is the basis of the familiar proposition of the *parallelogram of motions*. Imagine a body at A (Fig. 1) to have imparted to it simultaneously two velocities in the directions AB and AC, the magnitudes of which are represented by the lines AB and AC respectively. The body will move in the direction AD.

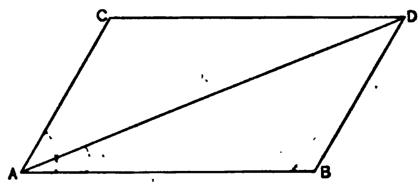

Fig. 1.

For, if we consider the motions to take place separately, the body, in obedience to the first velocity only, will move from A to B during one second; and, if it then moves in obedience to the second velocity only, it will move from B to D. If the same analysis is made for any part of a second, at the end of the two equal intervals of time the body will be at some point on AD, whose distance from A will bear the same relation to AD, that the interval of time chosen bears to the second. Hence, when the velocities are imparted simultaneously, the body will move in the direction

AD and the magnitude of the *resultant* velocity will be represented by the line AD.

According to the second law of motion, *forces are proportional to the velocities which they will impart to a given mass in the same time; and their lines of action must be in the same direction as the velocities produced.* Hence, in the parallelogram (Fig. 1), the lengths AB and AC may be taken to represent the magnitudes and directions of the forces which would impart the velocities in these directions to a unit mass, situated at A, in one second; and the length and direction of AD to represent the magnitude and direction of the resultant of these forces.

Another statement of this proof would be the following: If a unit mass were at rest at the point A, and two forces were applied simultaneously in the directions AB and AC, the accelerations produced would be a measure of the forces acting. But the paths described in equal times are proportional to the accelerations, and are in the directions of the forces acting; and may, therefore, be taken as a measure of the forces. Moreover, the paths described are proportional to the velocities produced in equal times. Hence, the diagonal AD will give the magnitude and direction of the resultant of the forces represented by AB and AC.

It is evident that this proof depends solely on the fact that forces are independent of each other in the effects they produce.

If we let R equal the resultant force represented by AD and F and F_1 equal the forces represented by AB and AC, respectively, and θ equal the angle BAC, the algebraic expression

$$R = \sqrt{F^2 + F_1^2 + 2\,FF_1 \cos\theta} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

will give us the magnitude of the resultant force.

If we let α equal the angle BAD, we have from the triangle BAD

$$F_1 : R = \sin\alpha : \sin\theta.$$

Hence
$$\sin\alpha = \frac{F_1 \sin\theta}{R}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and similarly,
$$\sin(\theta - \alpha) = \frac{F}{R}\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From these equations the direction of the resultant force may be found.

When $\theta = 90°$, equation (1) reduces to

$$R = \sqrt{F^2 + F_1^2} \quad \ldots \quad (4)$$

and

$$\sin \alpha = \frac{F_1}{R}, \quad \ldots \quad (5)$$

$$\cos \alpha = \frac{F}{R}. \quad \ldots \quad (6)$$

When $\theta = 0°$, equation (1) reduces to

$$R = \sqrt{F^2 + 2\,FF_1 + F_1^2} = F + F_1 \quad \ldots \quad (7)$$

and

$$\sin \alpha = \frac{F_1}{R} \sin \theta = 0. \quad \ldots \quad (8)$$

Hence $\alpha = 0°$.

This is the case of two forces acting along the same straight line. If they are equal and opposed they balance each other. Otherwise the resultant is equal to their algebraic sum and acts in the direction of the larger force.

Problems:

1. Given $F = 48.7$, $F_1 = 69.8$, $\theta = 65° 20'$; find R and α.
2. Given $F = 20.3$, $F_1 = 60.2$, $\theta = 135° 10'$; find R and α.
3. Given $F = 60.3$, $F_1 = 30.2$, $\theta = 90°$; find R and α.
4. Given $F = 26.7$, $F_1 = 45.7$, $\theta = 90°$; find R and α.

23. Resolution of Forces. — The resolution of a force into two components is the converse of finding the resultant of two forces acting at a point. If the lines, parallel to which the resolution is made, are oblique to each other, the components are called *oblique components*. Their magnitudes may be found by the solution of equations (2) and (3). (Art. 22.)

Very often it is desired to resolve a force into components parallel and at right angles to a given line. Such components are called *rectangular components*. In this case the magnitudes of the components may be found by the solution of equations (5) and (6). (Art. 22.)

These equations may be written

$$F = R \cos \alpha, \quad \ldots \quad (1)$$

$$F_1 = R \sin \alpha, \quad \ldots \quad (2)$$

giving F and F_1 as the rectangular components of R.

Problems:

1. $R = 140$,	$\alpha = 15°$,	$\theta = 35°$;	find F and F_1.
2. $R = 125$,	$\alpha = 25°$,	$\theta = 125°$;	find F and F_1.
3. $R = 325$,	$\alpha = 35°$,	$\theta = 90°$;	find F and F_1.
4. $R = 600$,	$\alpha = 55°$,	$\theta = 90°$;	find F and F_1.

α = angle R makes with horizontal.
θ = angle between components.
F = horizontal component.
F_1 = component making angle θ with F.

24. Triangle of Forces. — Since the resultant of two forces acting at a point is represented in magnitude and direction by the diagonal of a parallelogram constructed with its adjacent sides AB and AC equal in magnitude and parallel to the forces (Fig. 1), it is evident that the solution of the problem might be obtained by constructing the triangle ABD only.

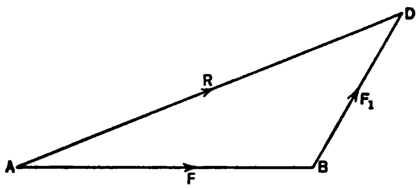

FIG. 2.

In this triangle the side AB may be considered to be a vector representing the force F, and BD a vector representing the force F_1. The side AD, obtained by adding these vectors (Fig. 2), would then be the vector representing the magnitude and direction of the resultant R.

Referring again to Fig. 1, it is evident, if a force were applied at A equal and opposite to the resultant R, that this force and the forces F and F_1 would balance. In that case, the sum of the vectors representing these forces would be zero. Another way of stating the proposition would be as follows:

Three forces, which, when simultaneously applied at a point, balance each other, can be correctly represented in magnitude and direction by the three sides of a triangle taken in order: and, conversely, if three forces can be represented in magnitude and direction, by the three sides of a triangle taken in order, then, if they are applied at a point they will balance each other.

25. Rigid Bodies. — A rigid body is one that does not undergo any alteration of shape when subjected to the action of external

forces. No body is absolutely rigid; but different bodies possess different degrees of rigidity, depending upon the material of which they are composed, and other circumstances.

We shall consider a body as rigid when, under the action of external forces, the change in the distances and directions of its different particles from each other is so small as to be negligible. When a force acts on a rigid body its effect, so far as motion is concerned, may be to produce translation only, or translation combined with a rotation.

26. Principle of the Rectilinear Transference of Force in Rigid Bodies. — If a force F were applied to a rigid body at the point A (Fig. 3), in the direction AB, whatever motion this force might produce would be prevented from taking place, if an equal and opposite force were applied at A, B, C or D, or at any point along the line of action of the force.

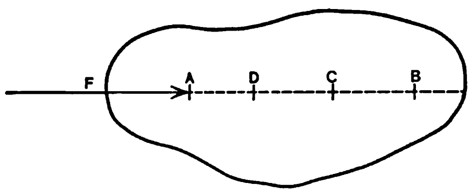

FIG. 3.

Since forces are independent of each other, in the effects they produce (Art. 13), changing the point of application of the force F, or the force balancing F, will have no effect on the resulting equilibrium, so long as the line of action remains the same. Changing the point of application, however, would alter the state of internal stress in the body, as will be shown later.

Hence we have the following principle: *The point of application of a force, acting on a rigid body, may be transferred to any other point, which lies in the line of action of the force and within the limits of the body, without altering the resulting motion of the body, or the equilibrium, if the force is one of a balanced system.*

27. Resultant of Two Forces in the Same Plane Acting at Different Points of a Rigid Body. — (a) *Forces not Parallel.* — Let two forces, F and F_1, in the same plane, act at the points A and B of a rigid body (Fig. 4a). To determine the resultant of these forces we may apply the principle deduced in Art. 26, and assume the point O, the intersection of their lines of action, as

the point of application of both forces. By means of the parallellogram of forces (Art. 22), we can obtain the magnitude and direction and line of action of the resultant force R, the point of application of R being any point on the line OC, within the limits of the body. It is evident that the same construction would hold even if the point O fell outside of the body. It is also

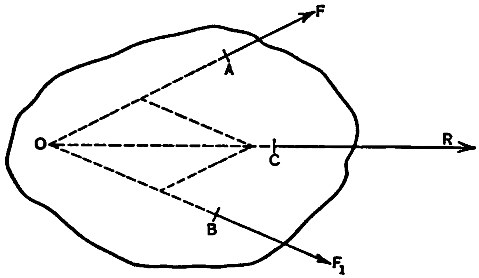

FIG. 4a.

evident that the magnitude and direction of the resultant of two nonparallel forces applied at different points in a rigid body, and acting in the same plane, may be obtained by constructing the triangle of forces; and its line of action, by drawing through the intersection of the forces a line parallel to the resultant vector of the triangle.

Since a force equal and opposite to R would balance F and F_1 it follows: *If three non-parallel forces acting at different points in a rigid body are in equilibrium,* (a) *they must act in the same plane;* (b) *their lines of action must pass through the same point;* (c) *their magnitudes and directions can be represented by the sides of a triangle taken in order.*

(b) *Parallel Forces.* — Let F and F_1 be two parallel forces acting at the points A and B (Fig. 4b). To determine the magnitude of the resultant we might treat this as a limiting case under non-parallel forces, the point of intersection, O, being at an infinite distance from A and B, whence $R = F + F_1$ (Art. 22).

By the following solution we can determine the position of the line of action of the resultant, as well as its magnitude. At A and B apply two equal and opposite forces, Aa and Bb, whose lines of action coincide with AB. These will balance and will not change the effect of the other forces.

Find the resultant Ae, of Aa and F, and the resultant Bd, of Bb and F_1, by constructing the parallelograms of forces. Then, by

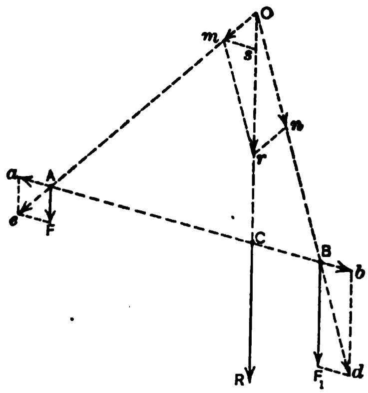

FIG. 4b.

constructing a parallelogram of forces at O, the intersection of Ae and Bd produced, we may find their resultant, Or, which is evidently the resultant of F and F_1. Draw ms parallel to AB. Then since Om is equal to and coincides with Ae and mr is equal and parallel to Bd, the triangles Oms and AeF are equal, and the triangles msr and BF_1d are also equal. Hence, the resultant, Or, is equal to $F + F_1$ and is parallel to the forces F and F_1; and may be represented by a force, $R = F + F_1$, applied at any point on a line through O parallel to the forces.

Let C be the intersection of R with the line AB. Then from similar triangles

$$\frac{AC}{OC} = \frac{Aa}{F} \quad \text{and} \quad \frac{BC}{OC} = \frac{Bb}{F_1}$$

Hence $$AC \times F = BC \times F_1$$

and $$\frac{F}{F_1} = \frac{BC}{AC}.$$

When the forces act in opposite directions and are unequal the construction is given in Fig. 4c, from which it is evident that

$$R = F - F_1 \quad \text{and} \quad \frac{F}{F_1} = \frac{BC}{AC}.$$

Hence, the resultant of any two parallel forces, acting in the same direction, or of two unequal parallel forces acting in opposite direc-

tions, is parallel to the forces and equal to their algebraic sum and cuts a line joining their points of application into segments, the lengths of which are inversely proportional to the magnitudes of the forces.

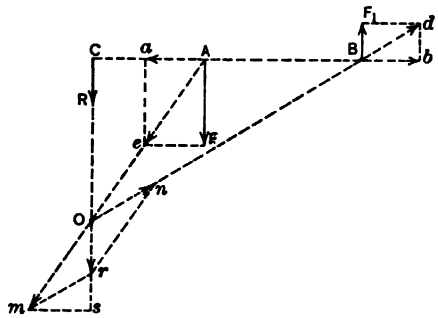

FIG. 4c.

Since in either case a force equal and opposite to R would balance F and F_1 it follows: *If three parallel forces are in equilibrium, they must act in the same plane; their algebraic sums must equal zero; and the segments, into which a straight line intersecting the forces is divided by their lines of action, will be inversely proportional to the magnitudes of the two outside forces.*

The case when the forces are parallel and equal and act in opposite directions will be considered later. Such a pair of forces is called a couple.

CHAPTER II.

STATICS.

§ 1. Preliminary.

28. System of Forces and its Resultant. — By a *system of forces*, we mean any number of forces which can be considered collectively. The *resultant* is the simplest equivalent system of forces, which can be substituted for the given system, so that the effect is the same in changing or preventing motion. The resultant of a system may be a single force, a couple, or a single force and a couple.

29. General Condition of Equilibrium of a System of Forces.— As stated in Art. 12, a system of forces is said to be in equilibrium, or balanced, when no change in motion is produced by its action on a body. In such a case, it is evident that the resultant of the system must be equal to zero.

30. Methods of Analysis. — In determining the *laws of equilibrium* of a given system of forces, we shall first find a method of obtaining the resultant of an unbalanced system of the same kind, and then deduce the conditions which must prevail when this resultant is equal to zero.

A part of the propositions in this chapter, will, therefore, be applicable in the treatment of the subject of Kinetics, as well as Statics.

The deduction of the laws of equilibrium will be followed with illustrations of their application in the solution of problems. We shall find that these laws enable us to use two general methods in the solution of problems in Statics; the *graphical* and the *analytical*, or *algebraical*. In this chapter we shall confine ourselves to the analytical method, and only in certain cases refer briefly to the graphical, leaving a more extended discussion of the latter until later.

31. Grouping of Force Systems. — For convenience, we shall consider force systems in the three following groups, in the order named: (1) Forces acting in a single plane, (2) Forces not acting

in the same plane, (3) Distributed forces. In dealing with the first two groups we shall consider the forces as concentrated (Art. 20).

§ 2. Forces Whose Lines of Action are Confined to a Single Plane.

32. Forces Acting through the Same Point. — Polygon of Forces. — If more than two forces act through the same point we may determine the resultant of the system as follows:

Find the resultant of two of them by the parallelogram, or by the triangle of forces; combine this resultant with the third force in the same way, and if there are more than three forces combine the second resultant with the fourth, repeating the process until the final resultant, which is evidently the resultant of the system, is obtained.

To state this in another way, let F, F_1, F_2, etc., represent a system of forces acting through O (Fig. 5a).

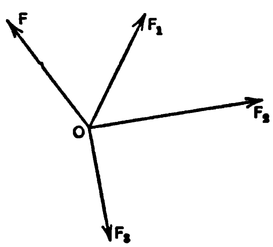

Fig. 5a.

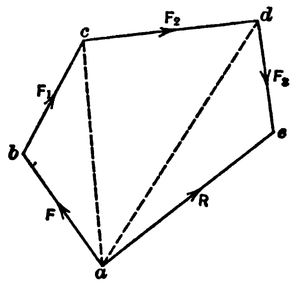

Fig. 5b.

Find the resultant of F and F_1 by adding their vectors ab and bc; the vector ac giving the magnitude and direction of their resultant (Fig. 5b). Add to this resultant the vector cd, representing the force F_2, and to ad, the sum of these, the vector de, representing the force F_3, thus obtaining ae as the vector which evidently gives the magnitude and direction of the resultant, R, of the system. A line drawn parallel to ae through O would then give the line of action of the resultant force. Hence, we may determine the magnitude and direction of the resultant of any system of forces acting through a point by determining graphically their vector sum; or, in other words, if the forces are represented by the sides of a polygon, taken in order, the magnitude and direc-

tion of the resultant will be represented by the closing side, taken in the opposite order.

If a force were applied at O, equal and opposite to the resultant, R, the system would be in equilibrium.

33. Graphical Conditions of Equilibrium. — The conditions deduced in Art. 32 may be stated as follows: *When a system of forces, whose lines of action are in the same plane and pass through the same point, is in equilibrium the vector sum of the forces is zero; or, they can be represented in magnitude and direction by the sides of a polygon taken in order.*

These may be called the *graphical conditions of equilibrium.*

In Art. 27, the conditions of equilibrium of three forces only have already been stated. The student should refer to them here.

34. Forces Acting through the Same Point. — Resolution of Forces. — Let F, F_1, F_2 represent any system of forces acting at the point O. It is required to find their resultant in magnitude and direction. With O as the origin refer the forces to a pair of rectangular coördinate axes, OX and OY (Fig. 6). Resolve each

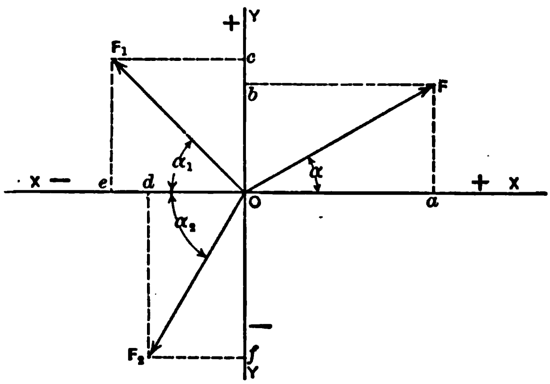

FIG. 6.

force into two components, one along OX and one along OY. The components of F will be Oa and Ob; of F_1, Oc and Oe; and of F_2, Od and Of. If α, α_1, α_2 represent the acute angles which F, F_1, F_2 make respectively with the axis OX, we have:

$$\begin{aligned} Oa &= F \cos\alpha, & Ob &= F \sin\alpha, \\ Oe &= F_1 \cos\alpha_1, & Oc &= F_1 \sin\alpha_1, \\ Od &= F_2 \cos\alpha_2, & Of &= F_2 \sin\alpha_2. \end{aligned}$$

If a component acts upward or toward the right we will assume it to be positive; if downward or toward the left, negative.

Then the algebraic sum of the components along OX will be $Oa - Od - Oe = \Sigma X$, and the algebraic sum of the components along OY will be $Ob + Oc - Of = \Sigma Y$.

Suppose ΣX to be negative and ΣY positive. Then lay off ΣX and ΣY on the axes OX and OY, in the directions indicated by the

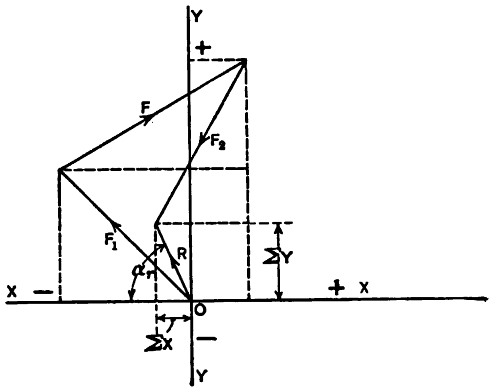

FIG. 7.

algebraic sign of the components (Fig. 7). The resultant of ΣX and ΣY and hence of F, F_1 and F_2 will be

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}. \quad \text{(Art. 22.)}$$

The direction of R may be obtained from the equation

$$\cos \alpha_r = \frac{\Sigma X}{R}, \quad \text{or} \quad \sin \alpha_r = \frac{\Sigma Y}{R}, \quad \text{or} \quad \tan \alpha_r = \frac{\Sigma Y}{\Sigma X},$$

the quadrant in which the resultant lies being determined by the directions of ΣX and ΣY along the coördinate axes.

In Fig. 7 the polygon of forces for the system is also constructed, showing graphically that ΣX and ΣY, the components of the resultant, are equal to the algebraic sums of the projections of the sides of the polygon on the two axes.

Usually, in engineering problems, the coördinate axes chosen are horizontal and vertical; in which case ΣH may be used to denote the sum of the horizontal, and ΣV the sum of the vertical components.

35. Algebraic Conditions of Equilibrium. — Since, in order that the forces of a system may balance each other, the resultant must be equal to zero, we determine from Art. 34 that when

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2} = 0,$$
$$\Sigma X = 0 \quad \text{and} \quad \Sigma Y = 0.$$

Hence the conditions of equilibrium may be stated as follows:

When a system of forces whose lines of action are in the same plane and pass through the same point is in equilibrium, if the forces are resolved into components along two axes at right angles to each other, the algebraic sum of the components in each of these two directions will be equal to zero.

These may be called the *algebraic conditions of equilibrium.*

36. Moment of a Force. — *The moment of a force, with respect to a point,* is the product of the force and the length of the perpendicular from the point to the line of action of the force. The perpendicular distance is called the *arm;* and the point, the *center of moments.*

The moment of a force, with respect to a line perpendicular to a plane containing the force, is the product of the force and the length of the perpendicular between the force and the line. The line is called the axis of moments.

The moment of the force F about the point O (Fig. 8) is $F \times OA$. It is evident that this is identical with the moment of the force about an axis through O, perpendicular to the plane containing O and the line of action of F.

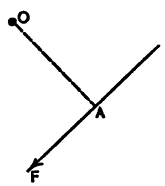

FIG. 8.

If a rigid body were supported on a fixed axis through O, but free to turn upon it, the force F would produce rotation about that axis and the tendency to rotate would be proportional to the moment of the force.

The unit moment is the moment of a unit force whose arm is a unit length. It will be called the foot-pound, inch-pound, meter-kilogram, etc., according to the units in which force and distance are expressed.

The sign of the moment is plus, if the tendency to rotate is clockwise or right-handed; and minus, if the tendency is counter-clockwise or left-handed.

37. Moment of the Resultant of Two Forces Applied at the Same Point. — Varignon's Theorem. — Let F and F_1 be the forces and R their resultant (Fig. 9). Let M be the trace of any axis perpendicular to the plane of the forces and let a, a_1, a_r be the moment arms of the forces with respect to M. Draw the coördinate

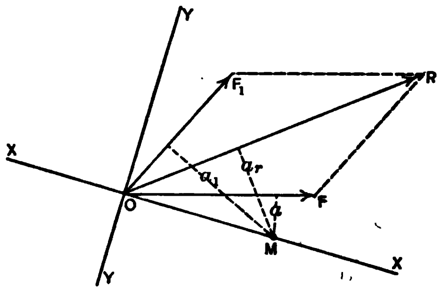

FIG. 9.

axis OX through M and let α, α_1, α_r be the respective angles between F, F_1, R and OX.

Then $\quad a = OM \sin \alpha;\ \ a_1 = OM \sin \alpha_1;\ \ a_r = OM \sin \alpha_r.$

From Art. 34,

$$\Sigma Y = F \sin \alpha + F_1 \sin \alpha_1 = R \sin \alpha_r.$$

Multiplying by OM

$$F\,(OM \sin \alpha) + F_1(OM \sin \alpha_1) = R\,(OM \sin \alpha_r).$$

Therefore, $$Fa + F_1a_1 = Ra_r.$$

If the center of moments M were taken between the lines of action of F and F_1, it is evident that the difference of the moments of F and F_1 would equal the moment of R. Moreover, if R were resolved into any two components, acting through any other point on its line of action, the moment of the resultant would be equal to the algebraic sum of the moments of its components.

Hence, we may state the following:

If a force be resolved into any two components, acting through any point on its line of action, the algebraic sum of the moments of these components with respect to any axis, perpendicular to their plane, will equal the moment of the original force about that axis.

Frequently it is easier to determine the moment of a force by computing the sum of the moments of its components than to determine it directly.

38. Moment of the Resultant of Any Number of Forces Acting through the Same Point. — Proceeding in the same manner as in Art. 37, we find that the algebraic sum of the moments of any system of forces, whose lines of action pass through the same point and lie in the same plane, with respect to any axis perpendicular to the plane of the forces, is equal to the moment of their resultant about that axis.

If the resultant is equal to zero the system is in equilibrium and we may state the following condition:

When any system of forces, whose lines of action pass through the same point and lie in the same plane, is in equilibrium the algebraic sum of their moments with respect to any axis, perpendicular to the plane of the forces, is equal to zero. This condition may be indicated by the expression $\Sigma M = 0$.

39. Summary of Conditions of Equilibrium for Forces whose Lines of Action Lie in the Same Plane and Pass through the Same Point. —

(1) Vector Sum = 0, or, closed polygon. (Art. 33.)

(2) $\left\{\begin{array}{lll} \Sigma X = 0, & \text{or,} & \Sigma H = 0. \\ \Sigma Y = 0, & \text{or,} & \Sigma V = 0. \end{array}\right\}$ (Art. 34.)

(3) $\Sigma M = 0$. (Art. 38.)

40. External vs. Internal Forces. — If a straight rod AB (Fig. 10) is in equilibrium under the action of a balanced system

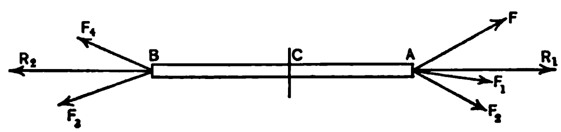

FIG. 10.

of external forces acting at the ends of the rod, as, for example, the forces F, F_1 and F_2 at A, which are balanced by the forces F_3 and F_4 at B, it is evident that, if we neglect the weight of the rod, the resultant, R_1, of the forces at A must be equal and opposite to the resultant, R_2, at B: and hence the line of action of R_1 and R_2 must be the line AB. Moreover, the rod itself will exert a reaction at A, equal and opposite to R_1, and an equal reaction at B, equal and opposite to R_2. The directions of the resultants, R_1 and R_2, of the forces shown in Fig. 10 are such that the rod is

subjected to a pull, or *tension*. If the forces acting were such that the directions of the resultants were reversed, the rod would be subjected to pressure, or *compression*.

If we consider a cross section through the rod at C; the part AC, on one side of this section, will exert a force at C on the part CB, which must be equal to the resultant, R_2, of the external forces acting at B; and an equal and opposite reaction will be exerted at C by the part BC on the part AC. This force on the section at C is called an *internal force*, or *stress*. It is a distributed force but, like the forces F, F_1, etc., and their resultants, R_1 and R_2, can be considered as a concentrated force acting along the middle of the rod.

Therefore, the stress on any cross section of a straight rod, which is in equilibrium under the action of external forces at its ends only, is equal in magnitude to the resultant of the forces acting at either end: and is tension, if that resultant exerts a pull on the rod, and compression, if it exerts a thrust.

A more complete discussion of stress will be taken up later, this definition being sufficient for our present needs.

41. Solution of Problems. — General Methods. — In general, problems in Statics are of the following kind: A system of forces is in equilibrium, part of the forces being known and some of them partly or wholly unknown; it is required to determine the magnitudes and directions, and possibly the lines of action, of the unknown forces.

In dealing with any problem involving the action of forces the first question which arises is: What are the forces to be dealt with and what are the elements to be determined?

After this question is answered correctly, the remainder of the analysis consists in applying the conditions of equilibrium best suited to the solution.

In answering the foregoing question it is suggested, that as an aid to correct analysis, the student make a sketch showing the lines of action and directions, as far as possible, of the forces to be dealt with, before applying the conditions of equilibrium. In many cases, by an inspection of the problem we may determine whether the stress in a member of a frame is tension, or compression, and hence the direction of the force the member will exert at a given point, before making a solution to determine the magnitude of the force. Wherever possible this should be done.

In the statement of problems, line sketches only will be used in the diagrams, and the student is expected to be able to give some idea of what the details of the different frames might be. Where several members meet a joint, it will always be assumed that the forces they exert act through the same point.

In each problem the weight of the frame will be neglected in making computations, unless otherwise stated.

42. Problems — Involving Balanced Systems of Forces Acting through the Same Point and Lying in the Same Plane. — In these problems the unknown elements may be determined by the application of the conditions of equilibrium given in the Summary in Art. 39. An analysis of these conditions shows us that we cannot determine more than two unknown elements in such a force system, and that we may determine these two by using any one of the conditions (1), (2) or (3). Hence we have three different methods of solving problems of this kind.

Problem 1.

The weight of 5000 lbs. suspended at A, is supported by the two members, AB and AC, attached to the wall at B and C. Find the stresses in the members AB and AC (Fig. 11).

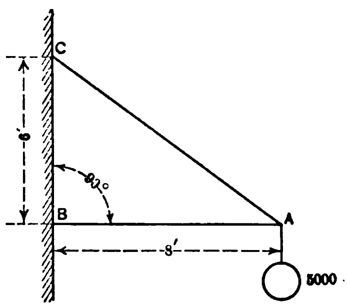

FIG. 11.

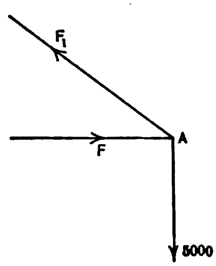

FIG. 12.

Solution. — In this problem we have a balanced system of three forces acting at A, viz., the weight of 5000 lbs. and the forces F and F_1, exerted by the members AB and AC, as shown in the sketch (Fig. 12). An inspection of the frame (Fig. 11) shows that the member AC is in tension and the member AB, in compression. Hence AC exerts a pull at the point A, and AB a thrust, the directions being indicated by the arrows (Fig. 12). The problem, therefore, is to determine the magnitudes of two unknown forces in a balanced system of three forces acting at a point, and the solution may be made in the three following ways:

First Method (Triangle of Forces). — The forces may be represented by the sides of a triangle taken in order, or, their vector sum equals zero (Art. 39).

Draw a vector representing the force of 5000 lbs. (Fig. 13) and, through its extremities, vectors to represent the forces F and F_1. If drawn to scale, the magnitudes of F and F_1 may be determined directly by measurement of the force triangle.

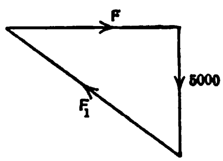

FIG. 13.

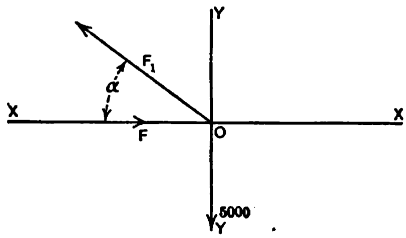

FIG. 14.

Instead of drawing the diagram to scale, the problem may be solved by observing that the triangle of forces will be similar to the triangle ABC (Fig. 11); and from the proportions between the homologous sides we have

$$5000 : F = 6 : 8,$$
$$5000 : F_1 = 6 : 10.$$

Solving these proportions: $\quad F = \frac{40{,}000}{6} = 6667$ lbs. (Compression);

$$F_1 = \frac{50{,}000}{6} = 8333 \text{ lbs. (Tension).}$$

Note. — The directions of the unknown forces are indicated by the force triangle, and can be determined from this whenever there is any difficulty in finding them by inspection, as previously suggested.

Second Method (Resolution of Forces). — This is the application of the second condition of equilibrium (Art. 39). Refer the forces to the coördinate axes OX and OY (Fig. 14).

Imposing the conditions of equilibrium and noting that

$$\sin\alpha = 0.6 \quad \text{and} \quad \cos\alpha = 0.8,$$

we have

$$\Sigma X = F - 0.8\,F_1 = 0,$$
$$\Sigma Y = 0.6\,F_1 - 5000 = 0.$$

The solution of these equations gives:

$$F = 6667 \text{ lbs. (Compression),}$$
$$F_1 = 8333 \text{ lbs. (Tension).}$$

Note. — In using this method if we are unable to determine the directions of the unknown forces by inspection, it will be necessary to assume them. Then if the solution gives positive values for the unknown forces it will indicate that the assumptions are correct. A negative value obtained for either of the unknown forces will indicate that the correct direction for the force is opposite to that assumed.

Third Method (Moments). — This is the application of the third condition of equilibrium (Art. 39). By choosing a moment axis, perpendicular to the plane of the forces, passing through some point in the line of action of one of the unknown forces, the moment of that force becomes zero, leaving only one unknown quantity in the moment equation.

If we take moments about B and note that the moment arm of the force F_1 will be 4.8 ft., we have

$$\Sigma M = 8 \times 5000 - 4.8\, F_1 = 0,$$
$$F_1 = 8333 \text{ lbs. (Tension).}$$

Taking moments about C we have

$$\Sigma M = 8 \times 5000 - 6\, F = 0,$$
$$F = 6667 \text{ lbs. (Compression).}$$

In this case, as in the preceding solution, the directions of the unknown forces might be assumed and their magnitudes determined, a positive result indicating a correct assumption for the direction of the unknown force.

In general, it is better to first find the moment of the known forces and observe its sign, whether plus or minus, then the direction of the unknown force must be such as to give a moment of the opposite sign.

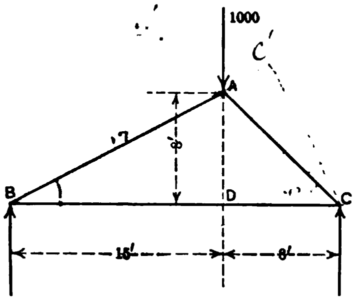

FIG. 15.

Problem 2.

Find the stresses in the members AB and AC, of the triangular frame ABC, the load at A being 1000 lbs. (Fig. 15).

Solution. — In this problem we have a balanced system of forces acting at A, consisting of the vertical force of 1000 lbs. and the unknown forces F and F_1 (Fig. 16). It is evident that F and F_1 are both compression, as indicated.

Draw the triangle of forces (Fig. 17) and solve analytically. From Fig. 15,

$$\angle BAD = 61° - 56'.$$

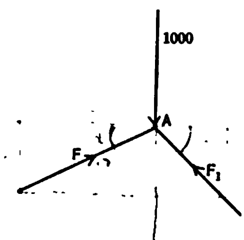

FIG. 16.

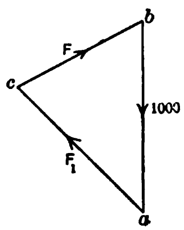

FIG. 17.

Hence in Fig. 17,

$$\angle a = 45°,$$
$$\angle b = 61° - 56'$$
$$\angle c = 73° - 4'.$$

Therefore,
$$1000 : F = \sin c : \sin a,$$
$$F = 739 \text{ lbs. (Compression)},$$
$$1000 : F_1 = \sin c : \sin b,$$
$$F_1 = 922 \text{ lbs. (Compression)}.$$

Problem 3.

Solve Problem 2, using method of *resolution of forces:* also by method of *moments.*

Problem 4.

Assuming that the frame in Problem 2 is supported by a vertical force at B, find the magnitude of the force and the stress in BC.

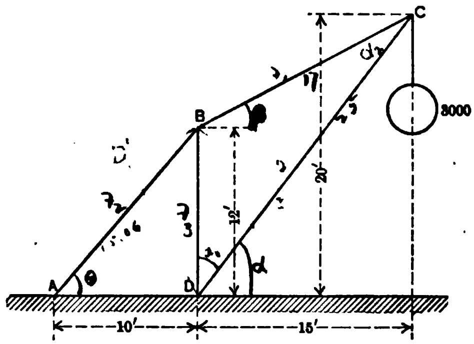

FIG. 18.

Problem 5.

The crane (Fig. 18) is loaded with 3000 lbs. at C. Determine the stresses in the boom CD, the tie BC, the mast BD and the stay AB.

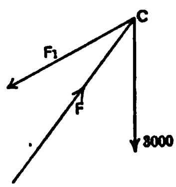

FIG. 19.

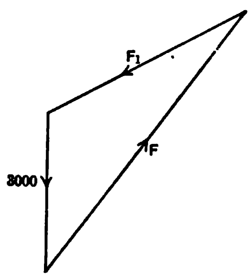

FIG. 20.

Solution. — At all three points B, C and D of the frame we have a balanced system of forces acting. To make the solution we must consider first a joint at which there are only two unknown forces. The joint C is evidently the only one fulfilling this condition, the forces acting being the weight of 3000 lbs.

and the unknown forces, F and F_1, exerted by the members DC and BC, respectively.

It is evident that the member CD is in compression and BC in tension, and hence the directions of the forces will be as shown in Fig. 19.

The solution of the force triangle for the point C (Fig. 20) may be made by noting its similarity to the triangle BCD (Fig. 18). The length of CD will be found by computation to be 25 ft. and the length of BC, 17 ft.

Hence,
$$F : 3000 = 25 : 12,$$
$$F = 6250 \text{ lbs. (Compression stress in } CD),$$
$$F_1 : 3000 = 17 : 12,$$
$$F_1 = 4250 \text{ lbs. (Tension stress in } BC).$$

Having found the stress in BC, we can determine the stresses in AB and BD by the triangle of forces for the point B. Let F_2 and F_3 be the forces exerted at the point B by the members AB and BD, respectively. Then it is evident that the stress in AB will be tension and that in BD, compression: and the directions of the forces will be as indicated in Fig. 21.

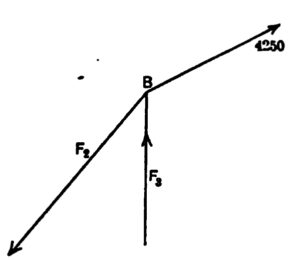

FIG. 21.

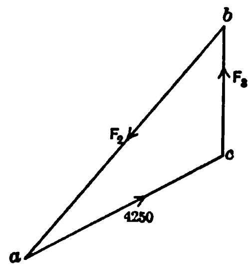

FIG. 22.

To solve the force triangle for the point B, we may determine the angles from the dimensions given in (Fig. 18).

$$\angle ABD = 39° - 48',$$
$$\angle CBD = 118° - 4'.$$

Then in the force triangle (Fig. 22)

$$\angle b = 39° - 48',$$
$$\angle c = 118° - 4',$$
$$\angle a = 22° - 8'.$$

Hence,
$$F_2 : 4250 = \sin c : \sin b,$$
$$F_2 = 5859 \text{ lbs. (Tension in } AB),$$
$$F_3 : 4250 = \sin a : \sin b,$$
$$F_3 = 2502 \text{ lbs. (Compression in } BD).$$

Problem 6.

Solve Problem 5, using method of *resolution of forces*.

Problem 7.

Find the horizontal and vertical components of the supporting forces at A and D, Problem 5.

Problem 8.

Find the stresses in the members of the crane given in Problem 5, when the boom makes an angle of 15° with the horizontal.

Problem 9.

Find the force F, necessary to keep the weight of 200 lbs. from sliding down a frictionless plane, as shown in Fig. 23.

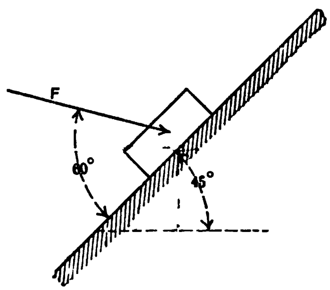

FIG. 23.

Problem 10.

If the magnitude of the force F, in Problem 9, was 200 lbs., what would be the angle between its line of action and the plane, necessary to prevent the weight from slipping down the plane.

Problem 11.

If the pressure P, exerted by the piston rod on the crosshead of a steam engine, is 1000 lbs. (Fig. 24), find the stress in the connecting rod AB, and the

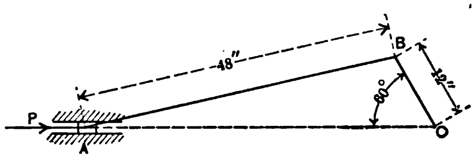

FIG. 24.

components perpendicular to and along the crank BO; assuming that the crank is prevented from turning and that there is no friction on the crosshead guides at A.

43. Couple. — Two parallel forces, equal in magnitude and opposite in direction, are called a couple. The perpendicular distance between the lines of action of the forces is called the *arm* of the couple; and the plane containing the forces is called the *plane* of the couple.

The moment of a couple is the algebraic sum of the moments of its forces about any axis perpendicular to its plane.

That the moment of a couple about any axis, perpendicular to its plane, is equal to the product of either force and the length of the arm may be shown by the following proof.

Let O be any axis, perpendicular to the plane of the couple (Fig. 25), and OA and OB, the moment arms of the forces with

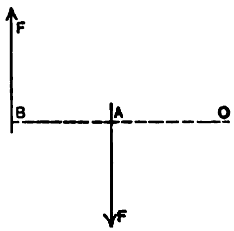

FIG. 25.

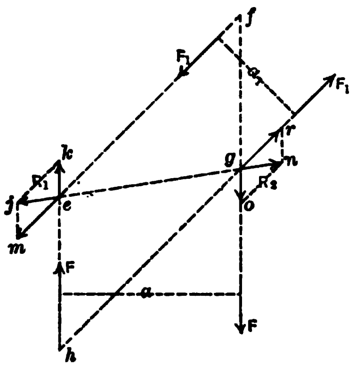

FIG. 26.

respect to O. Taking moments about O, we have $F \times OB - F \times OA = F \times AB$. It is evident that if the point O were taken between, or to the left of, the forces the sum of the moments would still be equal to $F \times AB$.

The unit moment of the couple is the same as that of a single force; namely, the *foot-pound, inch-pound,* etc., according to the units of force and distance which are used.

The sign of the couple is plus, if it tends to turn with right-handed rotation; and minus, if opposite.

44. Equilibrium of Two Couples. — *Proposition:* — *Two couples will balance when they act in the same plane and their moments are equal in magnitude, but of opposite sign.*

Proof. — Assume any two couples, Fa and $-F_1a_1$, in the same plane, with equal moments but opposite signs (Fig. 26).

Then numerically $$Fa = F_1a_1. \quad \quad (1)$$

Produce the lines of action of the forces, F and F_1, to form the parallelogram, *efgh*; and find the resultants of the two pairs of forces, F and F_1, by constructing the parallelograms of forces, *ekjm* and *grno*, at their points of intersection, e and g. It is evident that these parallelograms are equal and hence the magni-

tudes of the resultants, R_1 and R_2, of the two pairs of forces are equal; and their lines of action are either parallel, or they coincide.

The area of the parallelogram *efgh* is equal to

$$gf \times a = gh \times a_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Dividing equation (2) by equation (1) we have

$$\frac{gf}{F} = \frac{gh}{F_1} \quad \text{or} \quad \frac{gf}{ek} = \frac{gh}{em}.$$

Therefore, the parallelograms, *gfeh* and *ekjm*, are similar and the lines of action of R_1 and R_2 coincide and the couples are in equilibrium. Since no restrictions were imposed on the relative positions, or the lengths of the arms, of the couples in the preceding proposition the following will be true:

For any couple an equivalent couple may be substituted, situated anywhere in the same plane and having the same moment and sign, without changing the effect of the system of forces of which the couple is a part.

45. Resultant of Any Number of Couples Acting in the Same Plane. — Assume any system of couples acting in the same plane, such as F_1a_1, $-F_2a_2$, F_3a_3 (Fig. 27a). For these couples

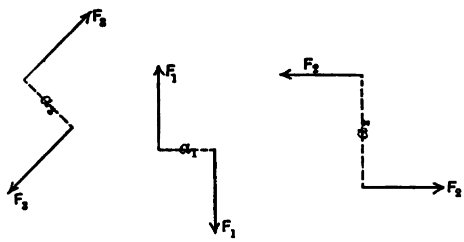

Fig. 27a.

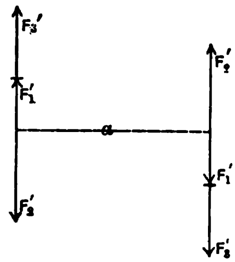

Fig. 27b.

we may substitute equivalent couples in the same plane (Art. 44), having a common arm a and placed so that the lines of action of the forces coincide. Let $F_1'a$, $-F_2'a$ and $F_3'a$ (Fig. 27b) be the couples which are equivalent to F_1a_1, $-F_2a_2$ and F_3a_3, respectively. It is evident that the resultant of the equivalent couples will be a couple whose forces are equal to $F_1' - F_2' + F_3'$ and arm is equal to a. Hence the moment of the resultant couple is

$$(F_1' - F_2' + F_3')\,a = F_1'a - F_2'a + F_3'a$$
$$= F_1a_1 - F_2a_2 + F_3a_3 = \Sigma Fa,$$

the last term in the equation being an abbreviation to represent the algebraic sum of the moments of the couples.

Therefore, *the resultant of any number of couples, acting in the same plane, is a couple whose moment is equal to the algebraic sum of their moments.*

If a system of couples acting in the same plane is in equilibrium the resultant must be equal to zero; hence,

If any number of couples, acting in the same plane, are in equilibrium the algebraic sum of their moments is equal to zero.

46. Resultant of a Single Force and a Couple. — Let F represent any force and F_1a_1, any couple acting in the same plane (Fig. 28a). For the couple F_1a_1 substitute an equivalent

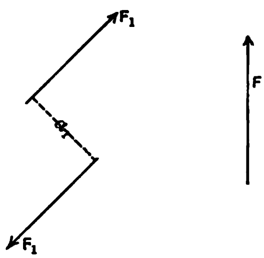

FIG. 28a.

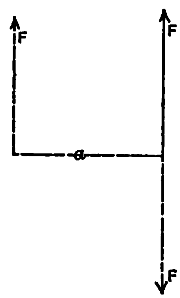

FIG. 28b.

couple (Art. 44) whose force is equal to the force F and arm equal to a, placing this couple so that one of its forces will balance the force F (Fig. 28b). The resultant will evidently be a force equal and parallel to F, at a perpendicular distance from it equal to the arm a, of the equivalent couple.

Since $$Fa = F_1a_1,$$

$$a = \frac{F_1a_1}{F}.$$

If F_1a_1 were negative the resultant force would evidently act at a distance a, to the right of F.

Therefore, the resultant of a single force and a couple, in the same plane, is a single force equal and parallel to the original force, having its line of action at a perpendicular distance from the original force equal to the moment of the couple divided by the force; and so situated

that the moment of the resultant about the point of application of the original force is of the same sign as the moment of the couple.

47. Resolution of a Single Force into an Equal and Parallel Force and a Couple. — This is the converse of the proposition given in Art. 46. It is evident that by applying at any point O two equal and opposite forces, equal and parallel to the force F (Fig. 29), we can resolve it into an equal and parallel force, and

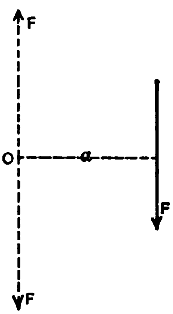

Fig. 29.

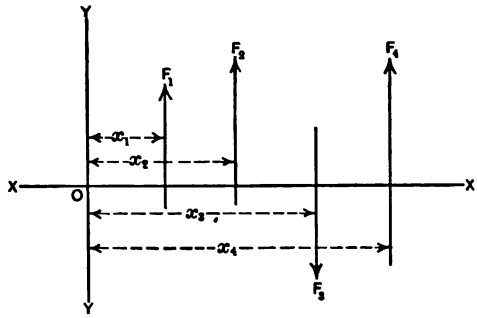

Fig. 30.

a couple whose moment will be equal to the moment of the force F with respect to the point O. In certain cases it will be found convenient to resolve forces in this way.

48. Composition of Parallel Forces. — Let F_1, F_2, F_3, F_4 represent any system of parallel forces acting in the same plane (Fig. 30). Refer the system to the coördinate axes, OX and OY, in the plane of the forces, assuming the axis of OX perpendicular to the forces.

Resolve each force into a component force acting along OY and a couple (Art. 47).

Let ΣF equal the algebraic sum of the components along OY and let ΣFx equal the algebraic sum of the moments of the component couples (Art. 45). In this way we may reduce the original system to a single component force, ΣF, acting along OY and a single component couple, ΣFx.

Three cases will now be considered.

Case I. — When $\Sigma F >< 0$ and $\Sigma Fx >< 0$.

Combining the force and the couple (Art. 46), the resultant of the system will be a single force,

$$R = \Sigma F,$$

and the distance of its line of action from O will be

$$x_r = \frac{\Sigma Fx}{\Sigma F}.$$

Therefore, $$Rx_r = \Sigma Fx;$$

that is, the moment of the resultant with respect to the origin is equal to the algebraic sum of the moments of the forces with respect to that point.

In determining whether x_r is measured to the right or left of O it is necessary to note that the sign of the moment of R about O must be the same as the sign of the couple ΣFx. If $\Sigma Fx = 0$, it is evident that the line of action of R coincides with OY.

CASE II. — When $\Sigma F = 0$ and $\Sigma Fx >< 0$.

In this case the resultant of the system is a couple whose moment is ΣFx, and the algebraic sum of the moments of the forces with respect to any point in their plane is the same as that with respect to any other point.

CASE III. — When $\Sigma F = 0$ and $\Sigma Fx = 0$.

In this case the system of forces is in equilibrium.

49. Conditions of Equilibrium of a System of Parallel Forces. — From Art. 48 we deduce the following conditions of equilibrium:

If a system of parallel forces acting in the same plane is in equilibrium, the algebraic sum of the forces is equal to zero, and the algebraic sum of the moments of the forces about any axis perpendicular to their plane is equal to zero.

It is evident that these conditions enable us to determine two unknown elements in a balanced system of parallel forces.

50. Problems. — Parallel Forces Acting in the Same Plane.—

Problem 1.

Find the resultant R of the parallel forces 200 lbs., 500 lbs. and 800 lbs., acting as shown (Fig. 31).

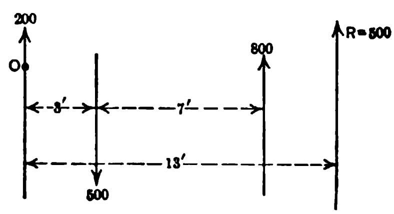

FIG. 31.

Solution. — Assume the axis OY, coinciding with the line of action of the force of 200 lbs., and the axis OX perpendicular to the forces.

Then $\Sigma F = 200 - 500 + 800 = 500$

and $\Sigma Fx = 500 \times 3 - 800 \times 10 = -6500.$

Hence $$x_r = \frac{\Sigma Fx}{\Sigma F} = \frac{6500}{500} = 13.$$

Therefore, $R = 500$ lbs. (acting upward), and the distance of its line of action from O is 13 ft. (to the right).

Note. — In computing the value of x_r we neglect algebraic signs and note that, since ΣFx is negative, the moment of R must be negative: hence its line of action must be to the right of O.

Problem 2.

Let AB represent a beam carrying the weights of 1000 lbs., 2000 lbs. and 3000 lbs. and supported by the vertical forces F_1 and F_2 (Fig. 32).

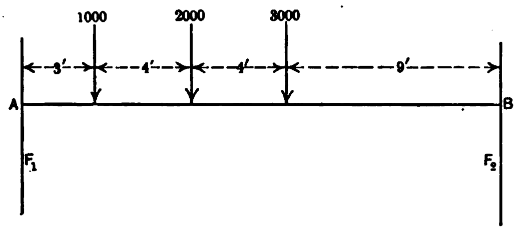

FIG. 32.

Solution. — In this case, we have a balanced system of parallel forces and by applying the conditions of equilibrium (Art. 49) we can determine the unknown forces. The sum of the moments of the forces with respect to the point A on the line of action of F_1 will be

$$\Sigma Fx = 1000 \times 3 + 2000 \times 7 + 3000 \times 11 - 20\,F_2 = 0,$$

and $F_2 = 2500$ lbs. (acting upward).

In a similar manner we may determine F_1, by taking moments about B,

$$\Sigma Fx = -3000 \times 9 - 2000 \times 13 - 1000 \times 17 + 20\,F_1 = 0,$$

and $F_1 = 3500$ lbs. (acting upward);

or, in a simpler way, we may determine F_1 from the condition $\Sigma F = 0$, in which case

$$\Sigma F = -1000 - 2000 - 3000 + 2500 + F_1 = 0,$$

and $F_1 = 3500$ lbs. (acting upward).

If both solutions are used we have a check on the numerical work.

Problem 3.

Determine the resultant of the system of parallel forces shown in Fig. 33.

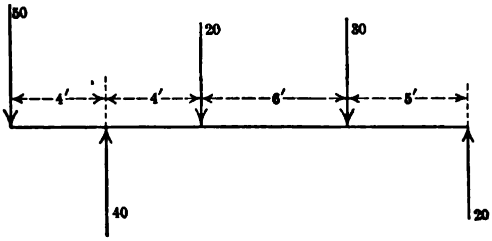

Fig. 33.

Problem 4.

Determine the resultant of the system of parallel forces shown in Fig. 34.

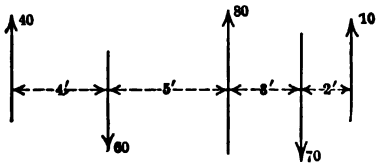

Fig. 34.

Problem 5.

Determine the resultant of the system of parallel forces and couples in Fig. 35.

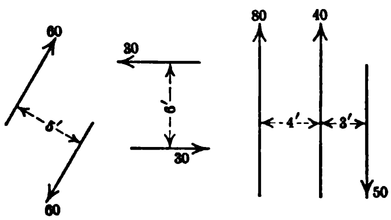

Fig. 35.

Problem 6.

Determine the resultant of the system of parallel forces and couples shown in Fig. 36.

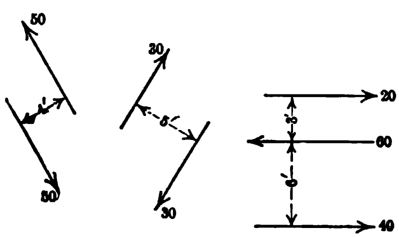

Fig. 36.

Problem 7.

The system of parallel forces shown in Fig. 37 is in equilibrium. Find the magnitudes and directions of the unknown forces F_1 and F_2.

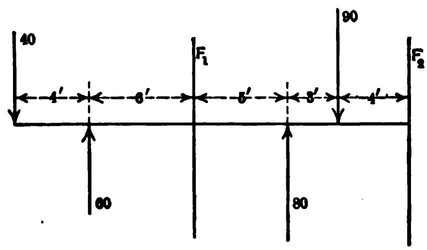

FIG. 37.

51. Resultant of any System of Forces Acting at Different Points and Situated in the Same Plane. — Let F, F_1 and F_2 be any system of forces whose lines of action are in the same plane,

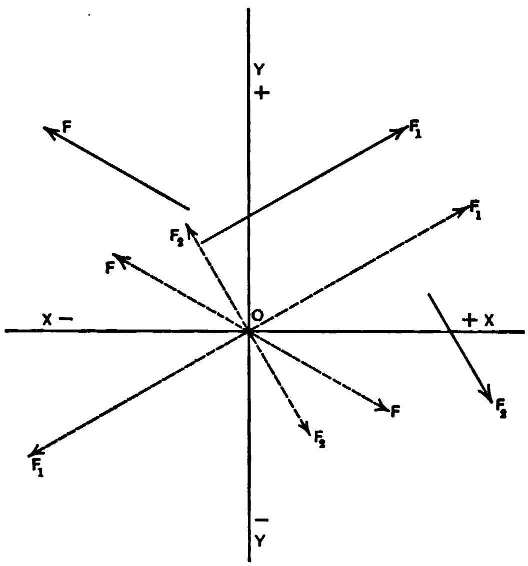

FIG. 38a.

but do not pass through the same point (Fig. 38a). Through any point O, in the plane of the forces, draw two axes at right angles, OX and OY. Resolve each force into a single force and a

couple by introducing at O equal and opposite forces, equal and parallel to the forces F, F_1 and F_2. (Art. 47.)

In this way the system may be resolved into an equivalent system of forces, F, F_1 and F_2, acting at O, which are equal, parallel to and act in same direction as original forces; and a system of couples, whose moments are equal to the moments of the forces about O.

The resultant of the system of couples is a single resultant couple (Art. 45) whose moment is

$$\Sigma M = \Sigma Fa.$$

The resultant of the system of forces acting at O may be found by the method of resolution of forces (Art. 34); in which case

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$$

and $$\cos \alpha_r = \frac{\Sigma X}{R}; \quad \sin \alpha_r = \frac{\Sigma Y}{R}; \quad \tan \alpha_r = \frac{\Sigma Y}{\Sigma X}:$$

or, the resultant may be found by determining the vector sum of the forces (polygon of forces) (Art. 32), as indicated in Fig. 38b.

Three cases will now be considered.

Case I. — When $R > 0$ and $\Sigma M >< 0$.

Combining the force R and the couple ΣM we get for the resultant of the system a force equal and parallel to R, whose line of action is at a perpendicular distance from O equal to $a_r = \frac{\Sigma M}{R}$ (Art. 46). The arm a_r must be laid off in such a manner that the sign of the moment Ra_r is the same as the sign of the moment of the resultant couple, ΣM (Fig. 38b).

It is evident that, if $\Sigma M = 0$ and $R > 0$, the resultant force acts through the origin.

Case II. — When $R = 0$ and $\Sigma M >< 0$.

In this case, the resultant of the system is a couple whose moment is equal to the algebraic sum of the moments of the forces about O; and hence the sum of the moments of the forces about any other point in their plane is the same as the sum of their moments about O.

Case III. — When $R = 0$ and $\Sigma M = 0$.

In this case, the forces are in equilibrium; and, since the component force acting at the origin is equal to zero, $\Sigma X = 0$ and $\Sigma Y = 0$, and the vector sum of the forces is equal to zero (Art. 39).

Note. — Since the moment of a force with respect to an axis is equal to the sum of the moments of its components (Art. 37), we might first resolve each force into components, parallel to OX and OY, acting through any point on its line of action, and then resolve these components into couples and forces, acting along OX and OY, and determine the resultant of the system as before. In the solution of problems it is frequently more convenient to use this method.

The preceding propositions, for determining the resultant of a system of forces acting at a point (Arts. 32 and 34) and a system

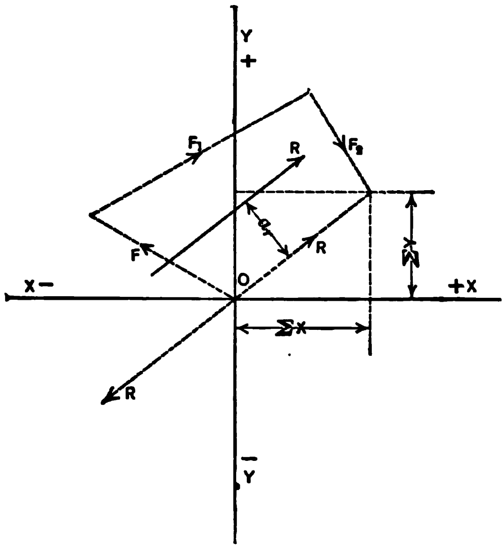

Fig. 38b.

of parallel forces (Art. 48), are evidently covered by this more comprehensive one, and could be considered as special cases under this general case.

52. Conditions of Equilibrium of any System of Forces Acting in the Same Plane. — The conditions of equilibrium, determined in Art. 51, may be summarized as follows:

$$\left.\begin{array}{l} \Sigma X = 0 \\ \Sigma Y = 0 \\ \Sigma M = 0 \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{l} \text{Vector sum} = 0. \\ \\ \qquad \Sigma M = 0. \end{array}\right.$$

These may be stated in the following manner: *If any system of forces whose lines of action are in the same plane is in equilibrium, the algebraic sum of their moments about any axis perpendicular*

to their plane is equal to zero and, if the forces are resolved into components in any two directions at right angles to each other, the algebraic sum of each set of components will equal zero; also, the vector sum of the forces will equal zero.

It is evident that these conditions will hold for a balanced system of parallel forces, or a balanced system of forces acting through a point, and that these two cases might be considered as special cases under the general one.

An analysis of these conditions shows us that the maximum number of unknown elements in a balanced system of forces, acting in the same plane, which can be determined by their use, is three and, if the forces are parallel or act through the same point, only two unknown elements can be determined. In case three nonparallel forces only are in equilibrium they must act through the same point (Art. 27).

53. Problems. — Involving Systems of Forces Acting in the Same Plane but not through the Same Point. — In the solution of these problems, the attention of the student is again called to the general methods of dealing with problems in Statics (Art. 41).

In nearly all cases, where a resolution of forces is made, horizontal and vertical axes are the most convenient to use. Hence, for the sake of brevity, we shall speak of the components as H components and V components and designate them with subscript letters to indicate the point of application, such as H_a, V_a, etc.

In determining certain unknown forces, such as the supporting forces of a frame, and certain actions and reactions at the joints it is usually sufficient to compute the H and V components only, the magnitudes and lines of action of the resultant forces acting at these points not being required.

In determining the internal forces acting at the different joints of a frame it will be necessary to apply the conditions of equilibrium to certain groups of forces. In picking out these groups for the purpose of solving any problem, we must keep in mind the fact that, if any frame is in equilibrium under a system of external forces, which would include loads and supporting forces, any member of the frame is in equilibrium under the forces acting upon it, which would include the forces exerted at the joints by which this member is connected to the rest of the frame.

Considering this member alone, the forces acting at the joints will constitute a balanced system of external forces, and their

lines of action and directions can be indicated on a sketch of the member. The conditions of equilibrium will enable us to determine the unknown elements in the system if they do not exceed the numbers specified for the different cases.

In applying the conditions of equilibrium, when three unknown forces are to be determined, we almost always may choose the axis of moments at the point of intersection of the lines of action of two of the unknown forces. Then the magnitude and direction of the third unknown force may be determined directly from the condition $\Sigma M = 0$. All three unknown forces may be found in this manner or, having found one, the remaining unknown forces may be determined by resolving the forces into H and V components, or by the polygon of forces.

Problem 1.

Determine the resultant of the system of forces, 30 lbs., 20 lbs., 40 lbs. and 60 lbs., acting as shown in Fig. 39.

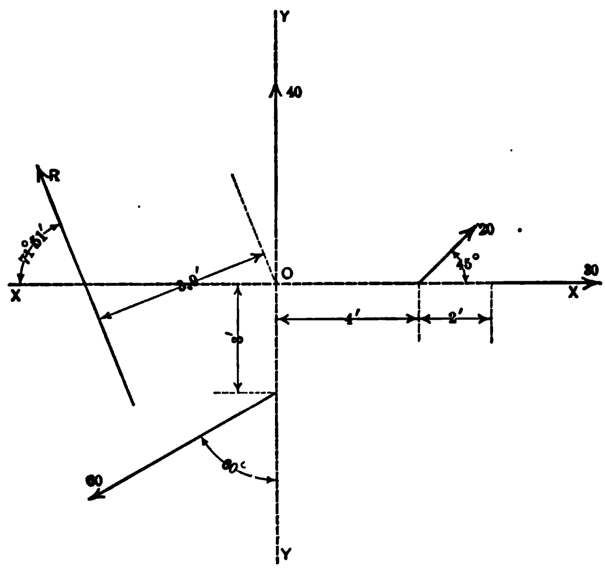

FIG. 39.

Solution. — Assume the coördinate axes OX and OY, coinciding with the lines of action of the forces of 30 lbs. and 40 lbs., assuming the positive directions as usual. Resolve the forces into components parallel to the axes and

determine ΣX, ΣY and ΣM, getting ΣM by adding the moments of the X and Y components.

Then
$$\Sigma X = 30 + 14.1 - 52.0 = -7.9,$$
$$\Sigma Y = 14.1 + 40 - 30 = 24.1,$$
$$\Sigma M = -14.1 \times 4 + 52 \times 3 = 99.6,$$
$$R = \sqrt{(24.1)^2 + (7.9)^2} = 25.4 \text{ lbs.},$$
$$\tan \alpha_r = \frac{24.1}{7.9}, \qquad \alpha_r = 71° - 51', \qquad a_r = \frac{99.6}{25.4} = 3.92 \text{ ft.}$$

An inspection of these results shows the line of action of the resultant to be that indicated in Fig. 39.

Problem 2.

Given the forces, 100 lbs., 200 lbs., 300 lbs. and 400 lbs., acting consecutively along the sides of a 4-ft. square. Determine their resultant.

Problem 3.

The frame shown (Fig. 40a) is subjected to a load of 2000 lbs. at C, and 1000 lbs. at D, and is supported at A and F in such a manner that the force acting on the frame at A is horizontal. Determine (*a*) the H and V components of the supporting forces at A and F; (*b*) the H and V components of the forces acting at the joints E, B and D and the stress in the member ED.

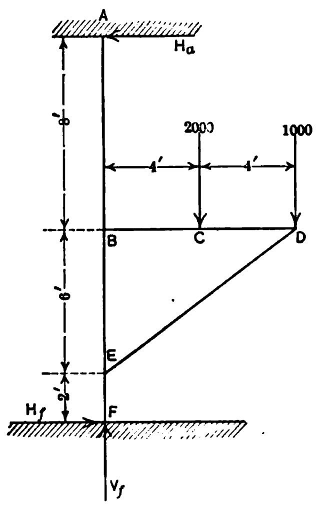

Fig. 40a.

Solution. — (a) We will first consider the whole frame as a rigid body. The external forces acting upon it will be the forces 2000 lbs., 1000 lbs., the horizontal force H_a, at A, and the components H_f and V_f of the supporting force at F.

These forces form a balanced system, in which we have three unknown elements to determine, and which may be found by applying the conditions of equilibrium (Art. 52). An inspection of Fig. 40a shows that the directions of the unknown forces will be as indicated.

It will be necessary to use the condition $\Sigma M = 0$ to determine one of these forces, and then the other two can be determined from the conditions $\Sigma H = 0$ and $\Sigma V = 0$. It is not necessary to show the axes of X and Y in the sketch.

Taking moments about the point F,
$$\Sigma M = 2000 \times 4 + 1000 \times 8 - 16 H_a = 0,$$
$$H_a = 1000 \text{ lbs.}$$

Then
$$\Sigma H = -1000 + H_f = 0,$$
$$H_f = 1000 \text{ lbs.},$$

and
$$\Sigma V = -2000 - 1000 + V_f = 0,$$
$$V_f = 3000 \text{ lbs.}$$

The directions of the forces determined from these equations agree with the directions already indicated for the unknown forces in Fig. 40a.

H_f and V_f might also be determined from the condition $\Sigma M = 0$, or by drawing a polygon of forces.

(b) We will now consider the member BD as a rigid body (Fig. 40b). The external forces acting upon it will be 2000 lbs., 1000 lbs., F, the force exerted at D by the member ED, and H_b and V_b, the components exerted at B by the vertical member AF.

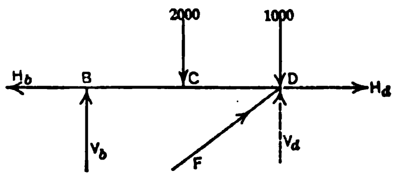

FIG. 40b.

Let H_d and V_d represent the H and V components of the force F.

Then
$$\frac{V_d}{H_d} = \frac{3}{4}.$$

The directions of the unknown forces will evidently be those indicated on the sketch.

The balanced system, therefore, contains three unknown elements. To determine these we will first find the sum of the moments about an axis through B.

$$\Sigma M = 2000 \times 4 + 1000 \times 8 - 8\,V_d = 0,$$
$$V_d = 2000 \text{ lbs.},$$

and
$$H_d = \frac{4}{3}\,V_d = 2667 \text{ lbs.}$$

Then
$$\Sigma H = 2667 - H_b = 0,$$
$$H_b = 2667 \text{ lbs.},$$

and
$$\Sigma V = 2000 - 1000 - 2000 + V_b = 0,$$
$$V_b = 1000 \text{ lbs.},$$

and the directions of the unknown forces agree with those indicated in Fig. 40b.

If we consider the member DE alone the forces acting will be H_d, V_d, H_e and V_e (Fig. 40c).

Hence $H_e = H_d = 2667$ lbs. and $V_e = V_d = 2000$ lbs., acting in the directions indicated. The member ED is in compression and the stress

$$F = \sqrt{(2667)^2 + (2000)^2} = 3333 \text{ lbs.}$$

The H and V components at B and E might be determined by applying the conditions of equilibrium to the forces acting on the member AF, thus getting a check on the previous solution. The system of forces for this case is indicated in Fig. 40d.

In each of the preceding cases, all the forces might be determined from the condition $\Sigma M = 0$, without using the conditions $\Sigma H = 0$ and $\Sigma V = 0$. In

each case, where the forces were determined by using the last two conditions, the polygon of forces might be used instead.

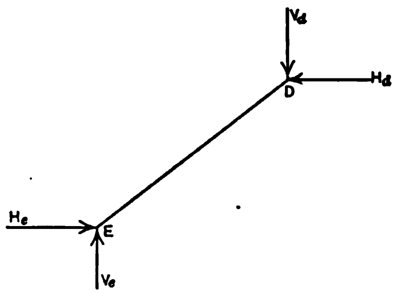

FIG. 40c.

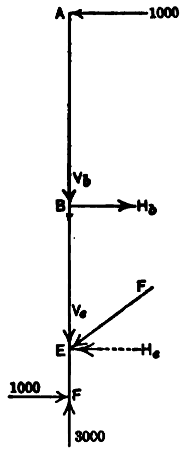

FIG. 40d.

The members BD and AF, which are acted upon by forces at other points besides the ends, are subjected to bending and we cannot determine the *stresses due to bending* from the principles thus far deduced.

Problem 4.

Determine the stress in BC and the H and V components of the force acting at A, in the frame fastened to the vertical wall (Fig. 41) and supporting weights of 600 lbs. and 800 lbs.

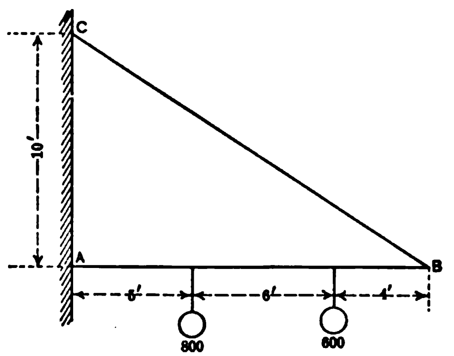

FIG. 41.

Problem 5.

Determine the H and V components of the supporting forces, assuming that the force at A is horizontal, and compute the H and V components of

the forces acting at the joints *C, D* and *E* of the crane (Fig. 42) when subjected to a load of 2000 lbs. at *F*.

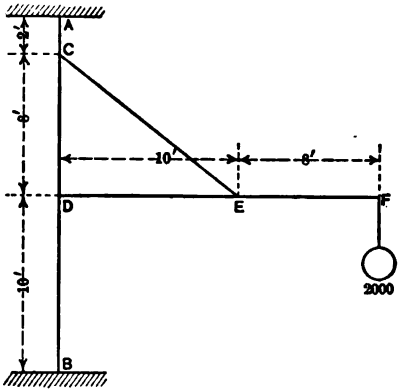

FIG. 42.

Problem 6.

The frame *ABC* (Fig. 43) is subjected to a load of 4000 lbs., perpendicular to the member *AB* at its middle point, and a vertical load of 8000 lbs. at *B*. Assuming that the supporting force at *A* is vertical, find its magnitude and the magnitudes and directions of the *H* and *V* components of the supporting force at *C*.

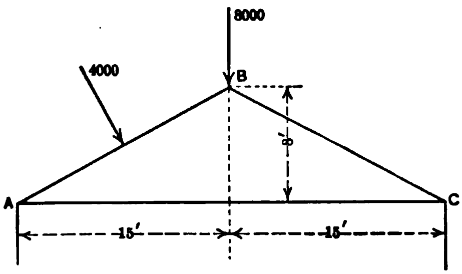

FIG. 43.

Problem 7.

Determine the stresses in the members *AC* and *BC* of the frame shown in Fig. 43.

Problem 8.

A homogeneous block of rectangular section weighing 5000 lbs. is held in the position shown (Fig. 44) by the force *P*. Find the magnitude of *P* and the friction between the weight and the plane necessary to prevent the edge *A* from slipping; also find the vertical pressure at *A*.

Problem 9.

The wheel weighing 600 lbs. (Fig. 45) is held in position by the horizontal force P, the friction between the wheel and the plane at A being sufficient to prevent slipping. Determine the force P, the force due to the friction at A and the normal pressure between the wheel and the plane.

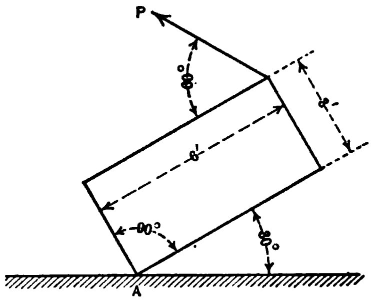

FIG. 44.

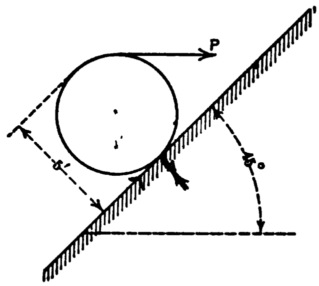

FIG. 45.

Problem 10.

The weight of 1000 lbs. is held in position by a flexible rope attached at A and running over the sheaves at D and A as shown (Fig. 46a), the force P applied to the end of the rope being parallel to AC. Find the stresses in the members AB and AC, assuming no friction at the axes of sheaves.

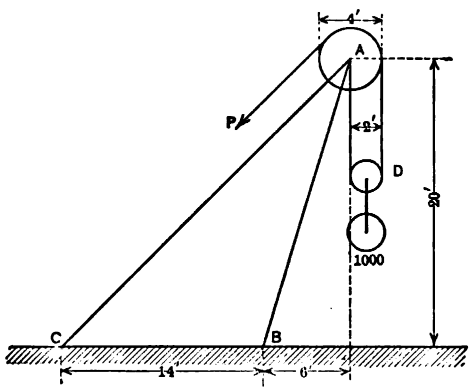

FIG. 46a.

Solution. — The force P will equal 500 lbs. The forces acting on the sheave A will be the two forces of 500 lbs. each, and the H and V components at the axis (Fig. 46b).

Hence, $H_a = 353$ lbs.,
$V_a = 853$ lbs.

We can now determine the stresses in AB and AC by applying the conditions of equilibrium to the forces acting through the point A (Fig. 46c). Assume the directions of the unknown forces F and F_1 as indicated. By resolution of forces into H and V components we have

$$\Sigma H = 0.287\,F - 0.707\,F_1 - 353 = 0,$$
$$\Sigma V = 0.958\,F - 0.707\,F_1 - 1353 = 0.$$

Solving: $F = 1490$ lbs. (Compression in AB).
$F_1 = 106$ lbs. (Tension in AC).

Since both F and F_1 are positive, the directions assumed were correct.

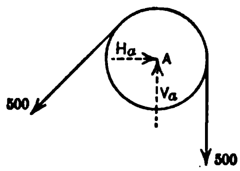

Fig. 46b.

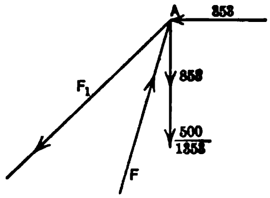

Fig. 46c.

Problem 11.

The weight of 2000 lbs. is balanced by the pull P on a rope leading over the sheaves A and B of the frame shown (Fig. 47). Find the H and V components at the supports C and D and at the joints E, F and G, assuming $V_c = 0$ and no friction at the sheaves A and B.

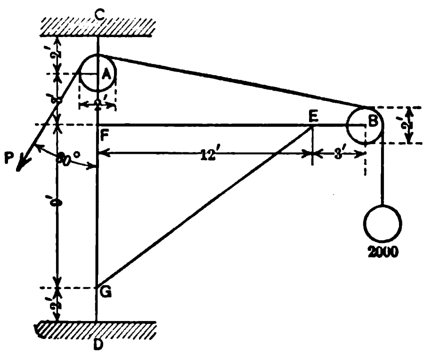

Fig. 47.

Problem 12.

A homogeneous bar AB, of uniform section, weighing 400 lbs., rests against the horizontal and vertical surfaces OA and OB and is held in position by a

horizontal tie *CD* (Fig. 48). Assuming that there is no friction at *A* and *B*, find the stress in *CD* and the normal pressure against the supporting surfaces at *A* and *B*.

Problem 13.

Solve Problem 12, assuming that the force exerted by the friction at *A* is 50 lbs. and at *B*, 20 lbs.

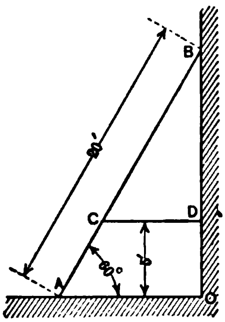

FIG. 48.

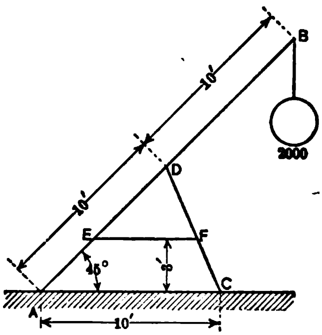

FIG. 49.

Problem 14.

The member *AB*, supporting a weight of 2000 lbs. (Fig. 49), is held in position by a pin at *A* and the brace *CD* and the tie *EF*. If the brace *CD* rests against a frictionless surface at *C*, find the stress in *EF* and the *H* and *V* components acting at the points *A*, *D* and *C*.

Problem 15.

Solve Problem 14, assuming that there is a force of 400 lbs. exerted by friction at the point *C*.

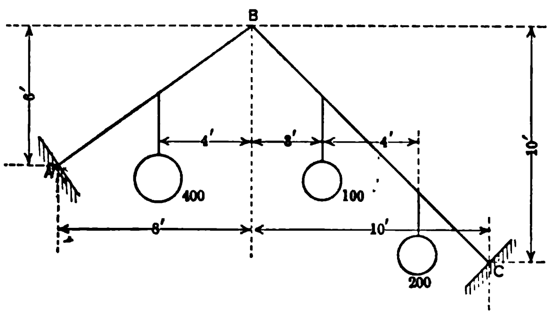

FIG. 50a.

Problem 16.

The bars *AB* and *BC* (Fig. 50a) are held by pins at *A*, *B* and *C* and carry the weights of 400 lbs., 100 lbs. and 200 lbs., as shown. Find the *H* and *V* components of the forces acting at the pins *A*, *B* and *C*.

Solution. — In this case we are unable to determine the supporting forces at A and C by applying conditions of equilibrium to the external forces acting on the whole frame since there are four unknown elements to be determined. The problem can be solved, however, by dealing with the H and V components of the forces acting on the two members separately, as follows:

Considering AB alone, the forces acting upon it are 400 lbs., H_a, V_a, H_b and V_b (Fig. 50b). If there is doubt in regard to the directions of the unknown components, these must be assumed.

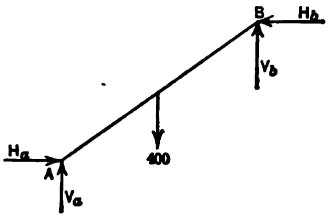

FIG. 50b.

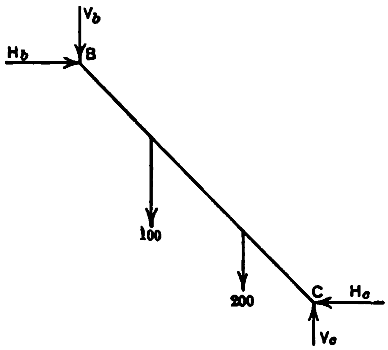

FIG. 50c.

In a similar manner the forces, acting on BC alone, may be indicated as shown (Fig. 50c). In assuming the directions of the components at B, the law of action and reaction must be kept in mind.

This analysis gives us two systems of balanced forces with four unknown elements in each, but, since the two unknown components, acting at B in one system, are equal and opposite to the two acting at B in the other, we have altogether only six unknown elements to determine; and these can evidently be found by applying the conditions of equilibrium to the forces acting in each system.

Hence, considering the forces acting on AB we have, taking a moment axis through A,

$$\Sigma M = 400 \times 4 - 6\,H_b - 8\,V_b = 0.$$

In the same manner for the forces acting on BC we have, taking a moment axis through C,

$$\Sigma M = -\,200 \times 3 - 100 \times 7 + 10\,H_b - 10\,V_b = 0.$$

The solution of these equations simultaneously gives:

$$H_b = 188.6 \text{ lbs.},$$
$$V_b = 58.6 \text{ lbs.}$$

Since the results are positive the directions assumed are correct. Then, by applying the conditions $\Sigma H = 0$ and $\Sigma V = 0$ to the forces acting on the members AB and BC, separately, we obtain

$$H_a = 188.6 \text{ lbs.}, \qquad H_c = 188.6 \text{ lbs.},$$
$$V_a = 341.4 \text{ lbs.}, \qquad V_c = 358.6 \text{ lbs.},$$

the directions being as indicated in Figs. 50b and 50c.

Problem 17.

Three bars, AB, BC and DE, are pinned together at D, B and E and supported on a frictionless surface at A (Fig. 51a). Find the supporting forces

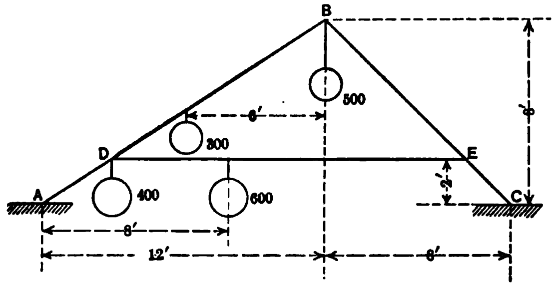

FIG. 51a.

at A and C, and the H and V components of the forces acting at the joints D, B and E, when the frame is subjected to the loads of 400 lbs., 600 lbs., 300 lbs. and 500 lbs., acting as shown.

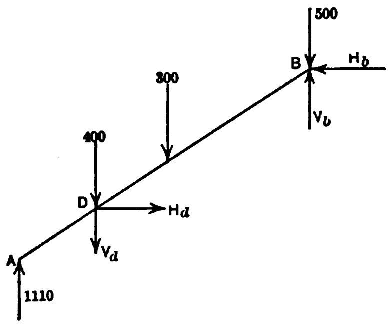

FIG. 51b.

Solution. — The supporting forces at A and C will be vertical and the forces acting on the whole frame form a balanced system of parallel forces. Taking an axis through A we have

$$\Sigma M = 400 \times 3 + 600 \times 8 + 300 \times 6 + 500 \times 12 - 20\,V_c = 0,$$
$$V_c = 690 \text{ lbs.},$$

and $\quad V_a = 400 + 600 + 300 + 500 - 690 = 1110$ lbs.

To obtain the components, acting at the joints, we must determine the forces acting on the members taken separately; and we will assume that the weights carried at D and B act on the member AB. Assuming the directions of the H and V components, keeping in mind the law of action and reaction we have the three balanced systems of forces shown in Figs. 51b, 51c and 51d. There are four unknown elements in each of the three systems, but the applica-

tion of the law of action and reaction reduces the entire number, to be computed, to six.

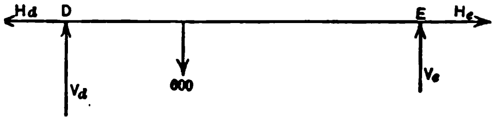

FIG. 51c.

The sum of the moments of the forces acting on AB (Fig. 51b), about an axis through D, will be

$$\Sigma M = 1110 \times 3 + 300 \times 3 + 500 \times 9 - 9\,V_b - 6\,H_b = 0,$$

and the sum of the moments of the forces acting on BC (Fig. 51d), about an axis through E, will be

$$\Sigma M = -\,690 \times 2 - 6\,V_b + 6\,H_b = 0.$$

Solving these equations we obtain

$$V_b = 490 \text{ lbs.},$$
$$H_b = 720 \text{ lbs.},$$

and the directions indicated in Figs. 51b and 51d are correct. By using the conditions $\Sigma H = 0$ and $\Sigma V = 0$, we may easily obtain the following forces, which act in the directions indicated in Figs. 51b, 51c and 51d.

$$H_d = 720 \text{ lbs.}, \qquad H_e = 720 \text{ lbs.},$$
$$V_d = 400 \text{ lbs.}, \qquad V_e = 200 \text{ lbs.}$$

The resultant vertical component acting at the point B, on the member AB, will be 10 lbs. (downward) and at the point D, 800 lbs. (downward).

Note. — In the solution of this problem the same values would be obtained for the H components and the resultant V components acting at the joints; if the force of 500 lbs. were considered as acting at B, on the member BC, and the 400 lbs., as acting at D, on the member DE.

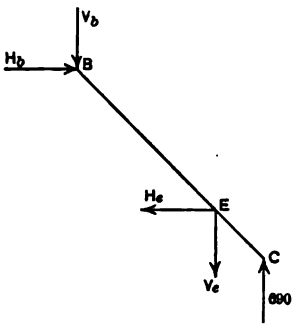

FIG. 51d.

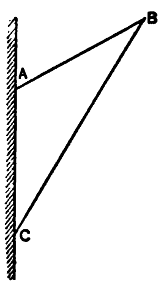

FIG. 52.

The solution of Problems 16 and 17 illustrates the point, that in certain cases we can determine four unknown elements in a system of forces, applied at different points in a rigid body and acting in the same plane, by applying

the law of action and reaction, together with the three conditions of equilibrium, to the forces acting on each of the members of the frame.

Problem 18.

Solve Problem 17, assuming that a bar joining A and C is put in to take the place of the member DE, and that a weight of 600 lbs. is applied at the middle of AC, still keeping weight of 400 lbs. at D.

Problem 19.

Prove that if two homogeneous rods, of uniform section and equal in weight are held in position by pins at A, B and C (Fig. 52), the line of action of the force exerted between the rods at B bisects AC.

Problem 20.

The bars AB and BC (Fig. 53) are held in position by pins at A, B and C, and subjected to a vertical force of 600 lbs. at B and the forces of 400 lbs. and 800 lbs., perpendicular to AB and BC at their middle points. Determine the H and V components of the forces acting at A, B and C.

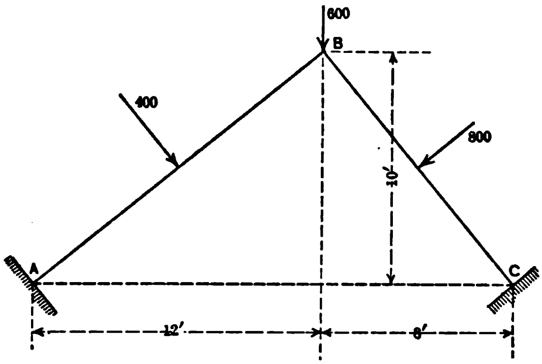

FIG. 53.

Problem 21.

The bars AB, BE and CD, supported on fixed pins at A and C and pinned together at B and E (Fig. 54), are acted upon by the weight of 100 lbs. and the

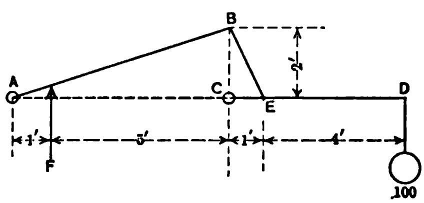

FIG. 54.

vertical force F. Find the magnitude of F, necessary to produce equilibrium, and the H and V components of the forces acting at A, B, C and E.

Problem 22.

The bars *AB*, *BC* and *CD*, supported on fixed pins at *A* and *D* and pinned together at *B* and *C* (Fig. 55), are acted upon by the weight of 100 lbs. and the horizontal force *F*. Find the magnitude of *F*, necessary to produce equilibrium, and the *H* and *V* components at *A* and the stresses in *BC* and *CD*.

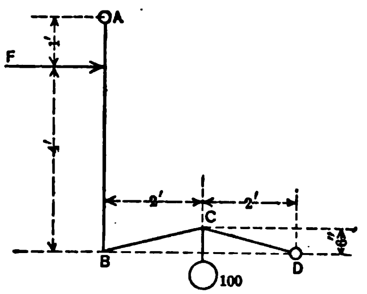

FIG. 55.

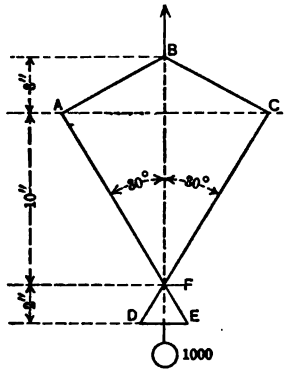

FIG. 56.

Problem 23.

The linkage made up of the bars *AB*, *BC*, *AE*, *CD* and *DE*, pinned together as shown (Fig. 56), is held by a vertical link at *B* and supports a weight of 1000 lbs. at the middle point of *DE*. Find the stresses in *AB* and *BC* and the *H* and *V* components of the forces acting at *D*, *F* and *C*.

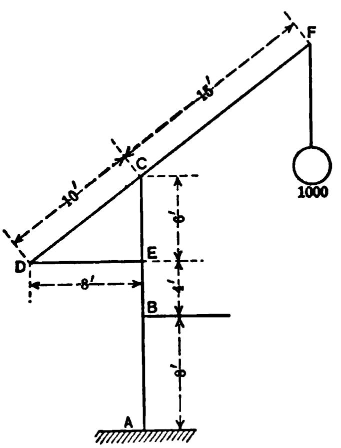

FIG. 57.

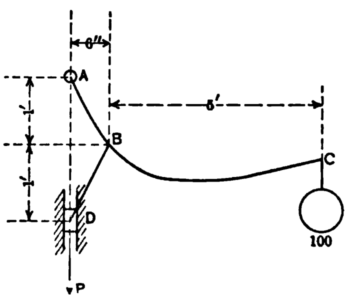

FIG. 58.

Problem 24.

The crane (Fig. 57) is supported on a pivot at *A* and held in the upright position by a horizontal force at *B*. Find the *H* and *V* components of the

supporting forces, the stress in DE and the H and V components of the force acting at C, due to the load of 1000 lbs. suspended at F.

Problem 25.

Find the counterweight necessary to apply at the point D (Fig. 57) to make the supporting force at B equal to zero. Calculate the supporting force at A, the stress in DE and the H and V components at C.

Problem 26.

The lever AC is supported on a fixed pin at A and the end D of the link BD works in a vertical slide at D (Fig. 58).

Find the vertical pressure P due to a weight of 100 lbs. at C, neglecting the friction of the parts. Find the H and V components acting at A, B and D.

Problem 27.

The two wheels, mounted on fixed axes at A and B, are connected by the link CD and are in equilibrium under the weights P, Q and W (Fig. 59). If

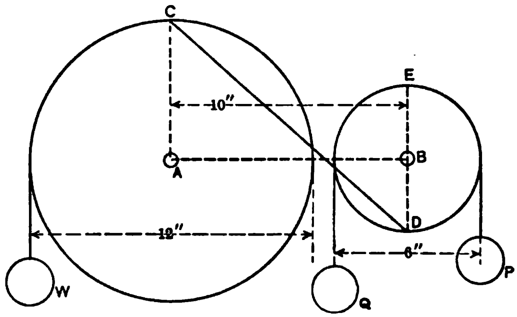

FIG. 59.

$P = 500$ lbs., $Q = 800$ lbs., find W, neglecting axle friction. Find the stress in CD and the H and V components at A and B.

Problem 28.

Solve Problem 27, assuming that the link CD is replaced by a link connecting C and E, and that $P = 1000$ lbs., $Q = 400$ lbs.

54. Method of Sections. — The method of sections is a method by which the internal stresses in the members of a frame may be conveniently determined. It consists in taking sections, cutting through certain members, and treating the part of the frame lying on one side of a section, or between two sections, as a rigid body.

The stresses at the cross sections of the different members cut by the section line, together with the external forces acting on the part of the frame in question, will form a balanced system. The

unknown elements in the system will be the stresses in the members cut by the section line; and, if these do not exceed the number that can be determined by imposing the conditions of equilibrium, the problem can be solved. Generally, this number will be three in the case of a frame whose members are in the same plane. In some cases, however, only two unknown stresses can be found, as in the case when a section is taken so that all three members, in which the forces are unknown, pass through the same point, or, if they should happen to be parallel to each other.

If the stresses in more than three members cut by a section are unknown, a solution cannot be made unless other independent conditions, in addition to the conditions of equilibrium, can be imposed.

55. Problems. — Method of Sections. –

Problem 1.

The truss shown (Fig. 60a) is subjected to a system of vertical loads, applied at the joints as indicated, so that the forces acting on all the members produce tension, or compression stresses, but no bending. Determine the stresses in the members cut by the sections XY and VW.

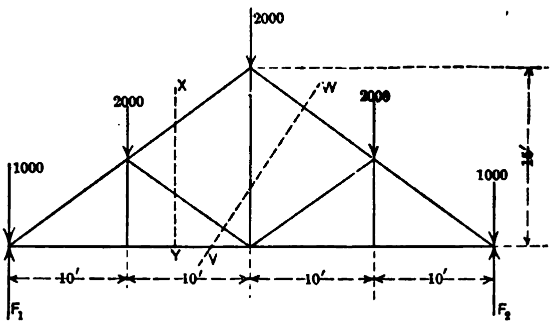

FIG. 60a.

Solution. — The supporting forces, F_1 and F_2, will be assumed vertical and hence equal to 4000 lbs. each.

If we take a section XY through the three members indicated, the part of the frame on one side of the section may be treated as a rigid body subjected to the action of a balanced system of forces, consisting of the known forces 4000 lbs., 1000 lbs. and 2000 lbs. (Fig. 60b) and the unknown forces F_3, F_4 and F_5, which are the stresses in the members cut by XY. The simplest solution, in this particular case, consists in using the condition of equilibrium, $\Sigma M = 0$, three times, taking the axis in each case at the intersection of the lines of action of two of the unknown forces.

Taking the axis through N,

$$\Sigma M = (4000 - 1000) \times 10 - 7.5\, F_3 = 0,$$
$$F_3 = 4000 \text{ lbs. (Tension.)}$$

Taking the axis through O,

$$\Sigma M = (4000 - 1000) \times 20 - 2000 \times 10 - 12\, F_4 = 0,$$
$$F_4 = 3333 \text{ lbs. (Compression.)}$$

Taking the axis through M,

$$\Sigma M = 2000 \times 10 - 12\, F_5 = 0,$$
$$F_5 = 1667. \text{ (Compression.)}$$

As in other problems in Statics, the directions of the unknown forces should be determined, as far as possible, by inspection; and verified from the sign of the moments of the unknown forces in the moment equations. In indicating the direction on the sketch (Fig. 60b), it should be remembered that the section, cut through a member by the section line XY, is to be regarded as the point of application of the force acting in that member. If the direction of the force in the member is towards the section, it is compression; and if away from it, tension.

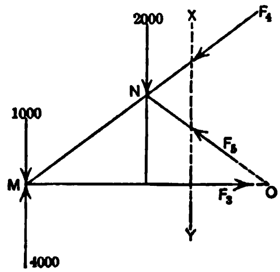

FIG. 60b.

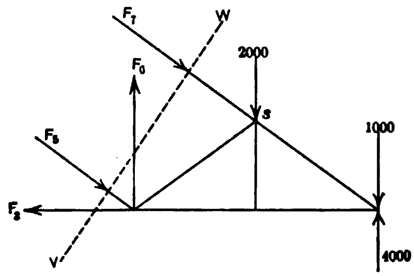

FIG. 60c.

The solution might be made by dealing with the forces acting on the part of the truss between the section and the right support, in which case the unknown forces would be the reactions, equal and opposite to F_3, F_4 and F_5. It is evident that in this case the simpler solution is the one involving the forces acting on the smaller part of the truss.

After having found one of the unknown forces by use of the moment equation, the remainder could be found by resolution of forces, or, by use of the polygon of forces. In cases where two of the unknown forces are parallel, the method of resolution of forces frequently gives the simplest solution.

By taking a section VW, we have the balanced system of forces shown in Fig. 60c. Having already determined F_3 and F_5, we can find the magnitude of the force F_6 by taking moments about an axis through s.

$$\Sigma M = -(4000 - 1000) \times 10 - 1667 \times 12 + 4000 \times 7.5 + 10\, F_6 = 0,$$
$$F_6 = 2000 \text{ lbs. (Tension.)}$$

It is evident that it would be impossible to determine F_6 until F_3, or F_5, had been determined by using the section XY, since the section WV cuts four members and three of them intersect at the same point.

It is also evident that $F_7 = F_4 = 3333$ lbs. (Compression.) By taking other sections, the stresses in all the members of the truss can be determined.

Problem 2.

The frame (Fig. 61) is held by fixed pins at A and B and supports the loads of 10,000 lbs. and 5000 lbs., at C and D as shown. Find the H and V com-

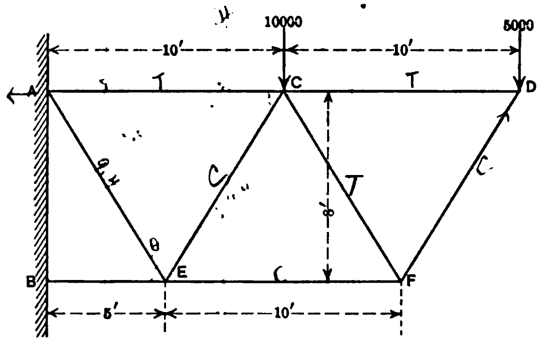

Fig. 61.

ponents of the supporting forces at A and B, and the stresses in the members of the frame by the *method of sections.*

Problem 3.

The frame (Fig. 62) is acted upon by the horizontal force of 5000 lbs. and the vertical forces of 6000 lbs. and 4000 lbs., and is fastened to a rigid support

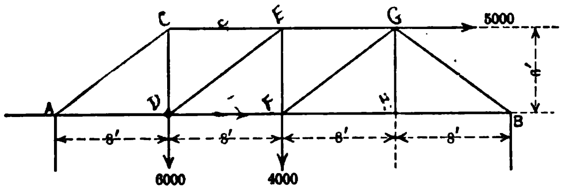

Fig. 62.

at A and supported by a vertical force at B. Find the H and V components acting at A and B and the stresses in the members by the *method of sections.*

Problem 4.

The frame (Fig 63) is supported on fixed pins at A and B and subjected to three horizontal forces of 1000 lbs. each and two vertical forces of 2000 lbs. each. Find the H and V components of the supporting forces and the stresses in the members of the frame by the *method of sections.*

Problem 5.

Two small trusses of the same dimensions are supported on fixed pins at A and C and pinned together at B (Fig. 64). Find the stresses in the members

due to the vertical load of 10,000 lbs. at B and the load of 8000 lbs., perpendicular to BC.

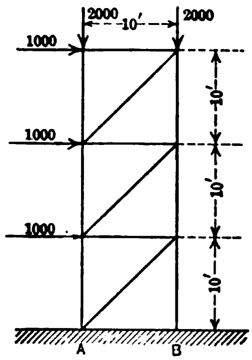

Fig. 63.

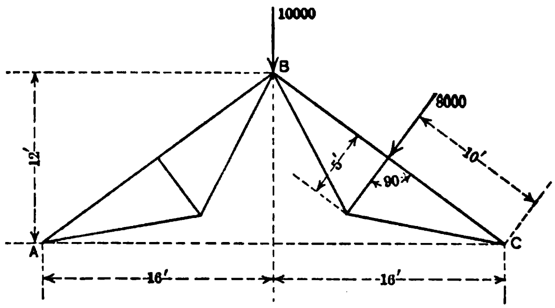

Fig. 64.

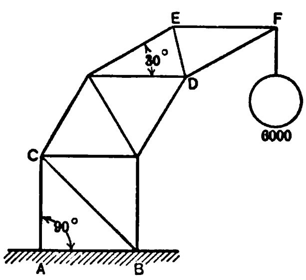

Fig. 65.

Problem 6.

The crane (Fig. 65) is supported at the points A and B and subjected to a load of 6000 lbs. at F. With the exception of DE and BC, the members are

all 10 ft. long. Find the stresses in the members and the H and V components acting at the supports A and B.

Note. — It is evident that, after determining the supporting forces, all the problems in this article might be solved by the methods used in solving the problems in Art. 42.

56. Statically Indeterminate Cases. — When, in any balanced system of forces acting on a rigid body, there are more unknown elements than can be determined by the methods of analysis, based on the principles of statics, the force system is said to be statically indeterminate. It is impossible to solve such a case except by making assumptions in regard to the relative magnitudes of certain of the unknown forces, and thereby reducing the unknown elements to the number that can be determined by applying the laws of equilibrium. The assumptions made in such cases will depend on the conditions under which certain of the unknown forces are exerted; and the correctness of the results obtained will evidently depend on how nearly the assumptions represent the actual conditions.

For example, if a straight rod is hung up by three parallel cords, the stresses in the cords and the force of gravity will form a balanced system of parallel forces, which is statically indeterminate. It is evident that one of the cords might be so slack that the entire weight would be carried on the other two, and the stresses in those two might be calculated on this assumption. Again we might estimate the proportion of the weight carried on one of the cords, and calculate the unknown forces on this assumption. The correctness of the results obtained in either case would evidently depend on the correctness of the estimate.

The following illustrations may be drawn from the problems in Art. 55.

If, in Problem 3, the frame were fastened to rigid supports at both A and B, it would be impossible to exactly determine the horizontal components of the supporting forces, and the stresses in part of the members of the frame. In order to solve such a problem, it is necessary to make an assumption in regard to the relative magnitudes of the horizontal components, and then the remaining unknown forces can be found. The assumptions most frequently made are: (a) the H components of the supporting forces are equal; or, (b) the H components of the supporting forces

are proportional to the vertical components. The correctness of the results will depend upon how nearly correct these assumptions are.

In Problem 4, if three additional diagonals were put in, running from the upper left-hand to the lower right-hand corner of each of the rectangular panels, the supporting forces and the stresses in the members would become statically indeterminate. An assumption might be made that the three additional diagonals were put in to brace the frame, when the horizontal forces act in the opposite direction, and that they would be under no stress when these forces act as indicated: in which case, the solution of the problem would evidently give the same results as when the extra diagonals were omitted.

In Problem 2, if a member BC were added to those already existing, it would be impossible to determine the components of the supporting forces and the stresses in BC, AC, AE, CE and BE. The stresses in CD, DF, CF and EF, however, would be the same as before. In this case the assumption might be made that the vertical components of the supporting forces at A and B would be equal. Then the stresses in all the members could be determined. It is evident that the correctness of the results obtained under this assumption would depend on the way in which the members AE and BC were tightened up, when the frame was put together, and also on the amount which the frame would yield under the forces acting upon it.

§ 3. Forces Whose Lines of Action are not Confined to a Single Plane.

57. Resolution of a Force into Components in Three Directions at Right Angles to Each Other. — Let $OE = F$ represent a force and OX, OY and OZ be any three coördinate axes at right angles to each other through O, which is any point on the line of action of F (Fig. 66). First resolve F into the two components, OC along the axis OZ, and OD perpendicular to OZ, and resolve OD into components along the axes OX and OY.

Then $$\overline{OE}^2 = \overline{OC}^2 + \overline{OD}^2 = \overline{OC}^2 + \overline{OA}^2 + \overline{OB}^2.$$

If we let the angle $XOE = \alpha$, $YOE = \beta$, $ZOE = \gamma$, it is evident that $OA = F \cos \alpha$, $OB = F \cos \beta$ and $OC = F \cos \gamma$. *Conversely,*

if OA, OB and OC represent any three components at right angles to each other, their resultant will be

$$F = \sqrt{\overline{OA}^2 + \overline{OB}^2 + \overline{OC}^2},$$

and $$\cos\alpha = \frac{OA}{F}, \quad \cos\beta = \frac{OB}{F}, \quad \cos\gamma = \frac{OC}{F}.$$

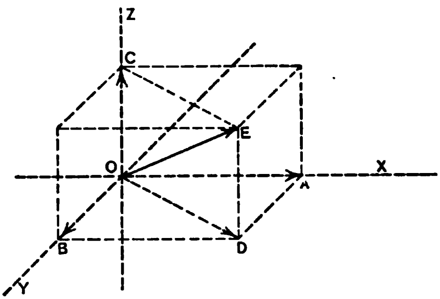

FIG. 66.

58. Composition of Forces Applied at the Same Point. — Let F, F_1, F_2 be any system of forces, acting at the same point, but not confined to the same plane (Fig. 67). Refer the forces to any

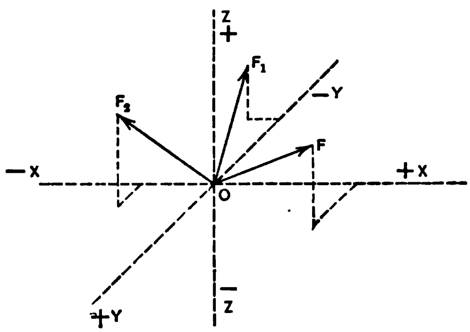

FIG. 67.

three axes at right angles to each other, taking the origin at their point of application, and let α, α_1, α_2, β, β_1, β_2 and γ, γ_1, γ_2 be the acute angles which the forces F, F_1, F_2 make with the axes OX, OY and OZ, respectively.

Resolve each force into components along the three axes (Art. 57), assuming the positive and negative directions as indicated.

Let ΣX, ΣY and ΣZ equal the algebraic sums of the components along OX, OY and OZ, respectively.

Then
$$\Sigma X = F\cos\alpha - F_1\cos\alpha_1 - F_2\cos\alpha_2,$$
$$\Sigma Y = F\cos\beta - F_1\cos\beta_1 + F_2\cos\beta_2,$$
$$\Sigma Z = F\cos\gamma + F_1\cos\gamma_1 + F_2\cos\gamma_2.$$

Combining these components the resultant of the system will be

$$R = \sqrt{\overline{\Sigma X}^2 + \overline{\Sigma Y}^2 + \overline{\Sigma Z}^2} \quad \text{(Art. 57)},$$

and if α_r, β_r, γ_r, are the angles which the resultant makes with the axes of OX, OY and OZ, respectively,

$$\cos\alpha_r = \frac{\Sigma X}{R}, \quad \cos\beta_r = \frac{\Sigma Y}{R}, \quad \cos\gamma_r = \frac{\Sigma Z}{R}.$$

To determine the quadrant in which the line of action of the resultant lies, it is necessary to note whether ΣX, ΣY and ΣZ are positive or negative, and to lay them off in the proper directions along the axes. The lines OA, OB and OC (Fig. 66) may be considered to be equal to ΣX, ΣY and ΣZ, respectively, and OE to be the resultant of the system, when all three components are positive.

Polygon of Forces. — The resultant might be found by adding the vectors representing the forces. The sides of the polygon formed in this case would not be in the same plane, the figure being a *gauche polygon.*

59. Conditions of Equilibrium for any System of Forces Acting at a Point. — It is evident from Art. 58 that when $R = 0$ its components must be zero, or

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0.$$

Therefore, *if any system of forces, whose lines of action pass through the same point, is in equilibrium, the algebraic sum of the components of the forces in each of three directions, at right angles to each other, must be equal to zero; also, the vector sum of the forces equals zero.*

Three unknown elements in such a system of forces can evidently be obtained by the use of these conditions.

60. Problems. — Forces Acting through the Same Point but not in the Same Plane. —

Problem 1.

Find the resultant of the system of forces, 80 lbs., 50 lbs., 40 lbs. and 60 lbs., acting at O (Fig. 68).

Problem 2.

A weight of 400 lbs. is suspended by three cords, each 6 ft. long, attached to the ceiling at points on a circle of 4 ft. in diameter, the radii from the points of attachment to the center of the circle making angles of 120° with each other. Find the tension in the cords.

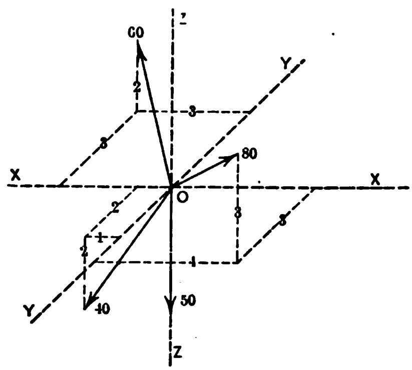

FIG. 68.

Problem 3.

The length of each of the shear legs, AO and OC (Fig. 69a), is 40 ft. Find the stress in the tie, OD, and the legs, OA and OC, due to a weight of 4000 lbs. suspended at O.

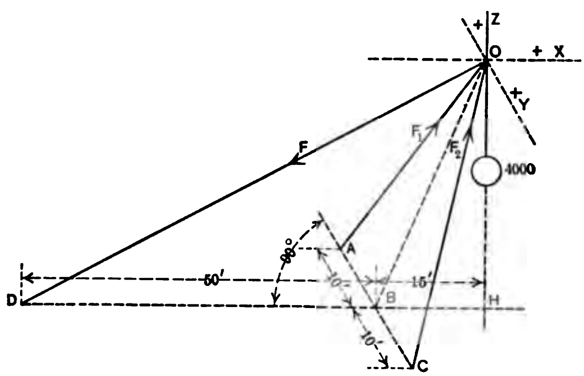

FIG. 69a.

First Solution. — The problem may be solved by resolving the forces into components, along three coördinate axes through O, and applying the conditions of equilibrium (Art. 59).

Calculate the dimensions not given as follows:

$$BO = \sqrt{(40)^2 - (10)^2} = \sqrt{1500} = 38.7 \text{ ft.}$$
$$OH = \sqrt{1500 - (15)^2} = \sqrt{1275} = 35.7 \text{ ft,}$$
$$OD = \sqrt{(65)^2 + 1275} = \sqrt{5500} = 74.2 \text{ ft.}$$

Assume the axis OZ in the line OH; the axis OX parallel to DH; the axis OY parallel to AC.

Assume the directions of the unknown forces, F, F_1 and F_2, as indicated.

The component of 4000 lbs. along $OZ = -4000$; along $OX = 0$; along $OY = 0$.

The component of F

$$\text{along } OZ = -F\frac{OH}{OD} = -F\frac{35.7}{74.2} = -0.48\,F;$$

$$\text{along } OX = -F\frac{DH}{OD} = -F\frac{65}{74.2} = -0.88\,F;$$

$$\text{along } OY = 0.$$

The component of F_1

$$\text{along } OY = F_1\frac{AB}{AO} = F_1\frac{10}{40} = 0.25\,F_1;$$

$$\text{along } BO = F_1\frac{BO}{AO} = F_1\frac{38.7}{40} = 0.97\,F_1;$$

$$\text{along } OX = F_1\frac{BO}{AO}\times\frac{BH}{OB} = F_1\frac{15}{40} = 0.38\,F_1;$$

$$\text{along } OZ = F_1\frac{BO}{AO}\times\frac{OH}{OB} = F_1\frac{35.7}{40} = 0.89\,F_1.$$

The component of F_2 along $OY = -0.25\,F_2$; along $OX = 0.38\,F_2$; along $OZ = 0.89\,F_2$.

Therefore

$$\Sigma X = -0.88\,F + 0.38\,F_1 + 0.38\,F_2 = 0,$$
$$\Sigma Y = 0.25\,F_1 - 25\,F_2 = 0,$$
$$\Sigma Z = -0.48\,F + 0.89\,F_1 + 0.89\,F_2 - 4000 = 0.$$

Solving these equations we obtain

$F = 2500$ lbs. (Tension).
$F_1 = 2900$ lbs. (Compression).
$F_2 = 2900$ lbs. (Compression).

The values obtained for F, F_1 and F_2, being positive quantities, show that the directions assumed for the unknown forces were correct.

Second Solution. — The simpler and better way of solving a problem of this kind is the following:

Divide the four forces acting through O into pairs; then the resultant of one of the pairs of forces must balance the resultant of the other pair. Moreover, since the resultant of two forces must act in the plane of the forces, the line of action of the two resultants must be the line of intersection of the two planes, containing the respective pairs. Hence the resultant of the forces 4000 lbs. and F will balance the resultant of the forces F_1 and F_2 and its line of action will be along BO. Let R be the resultant of the forces F_1 and F_2. Then the forces 4000, F and R are in equilibrium. Drawing the triangle of forces $a\,b\,c$ (Fig. 69b), and computing the proportionate lengths of the sides, by comparing the right triangles, abd and acd, with the similar triangles, BOH and DOH (Fig. 69a), we have $ab = 38.7$, $bc = 27.5$, $ac = 17.1$.

Then
$$\frac{F}{4000} = \frac{17.1}{27.5},$$
$$F = 2500 \text{ lbs. (Tension)},$$
and
$$\frac{R}{4000} = \frac{38.7}{27.5},$$
$$R = 5630 \text{ lbs.}$$

Resolve R into its components, F_1 and F_2, by drawing the triangle of forces (Fig. 69c). In drawing the triangle of forces, imagine that the plane AOC (Fig. 69a) of the forces, F_1 and F_2, is revolved into the plane of the paper. Divide the triangle abc into right triangles, as indicated, and compute the proportionate lengths of the sides by comparing with the similar triangles. AOB and BOC (Fig. 69a).

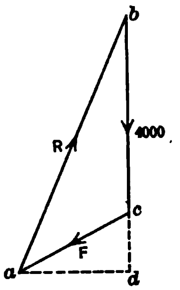

FIG. 69b.

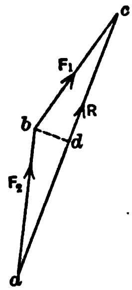

FIG. 69c.

Then
$$ab = bc = 40; \quad ac = 2 \times 38.7 = 77.4,$$
and
$$\frac{F_1}{5630} = \frac{40}{77.4}.$$
Hence
$$F_1 = 2900 \text{ lbs. (Compression)},$$
and
$$F_2 = 2900 \text{ lbs. (Compression)}.$$

It is evident that the method of resolution of forces, as applied to a system of forces acting in the same plane, might be used instead of the triangle of forces in this solution of the problem; also, the unknown forces in the two groups might be found by using the condition $\Sigma M = 0$.

Problem 4.

The member DC is supported on a vertical wall by a pin at D and the two equal ties, BC and AC, each 10 ft. long (Fig. 70). Find stresses in all three members due to load of 3000 lbs. supported at C.

Problem 5.

A tripod with three equal legs, each 8 ft. long, is supported on a horizontal plane with the feet at the vertices of an isosceles triangle whose sides are respectively 8 ft., 8 ft. and 5 ft. long. Find the stresses in the legs of the tripod if a load of 800 lbs. is applied at the top.

Problem 6.

The bar AB, perpendicular to a vertical wall (Fig. 71), is supported on a pin at B and held in position by the stays, AC and AD, fastened to the wall at the points C and D, which are on a horizontal line 10 ft. above B. The horizontal distances of C and D from a vertical plane through AB are respectively 4 ft. and 8 ft. Find the stresses in AC and AD and the H and V components of the supporting force at B, due to the weight of 600 lbs. supported by AB.

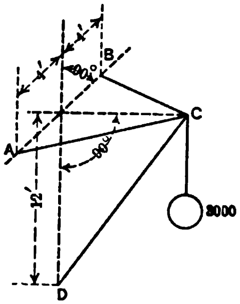

FIG. 70.

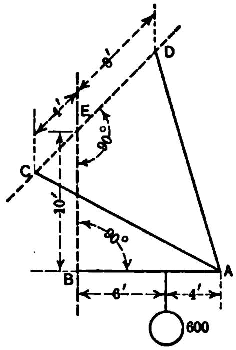

FIG. 71.

Problem 7.

The mast AB, of the crane (Fig. 72), is held in a vertical position by the stays BC and BD, fastened at the points C and D, in a horizontal plane. The vertical

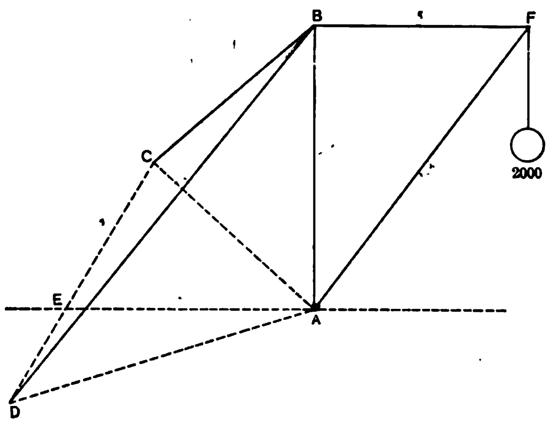

FIG. 72.

plane through ABF intersects the horizontal plane through A, in the line AE; and CD is perpendicular to AE. $AB = 20$ ft., $BF = 15$ ft., $AF = 25$ ft., $AE =$

20 ft., $CE = 12$ ft. and $DE = 8$ ft. Find the stresses in all the members of the frame due to the weight of 2000 lbs. supported at F.

61. Moment of a Force with Respect to Any Line. — If a force be resolved into two components, one parallel to and the other in a plane perpendicular to the given line, the moment of the latter component with respect to the line (Art. 36) is called the moment of the force with respect to the line.

It can easily be shown that, if a force be resolved in the manner indicated, the moment arm of the component in the plane perpendicular to the line is the same, no matter at what point on the line of action of the original force the resolution is made. Let the plane of the paper be any plane, perpendicular to an axis of

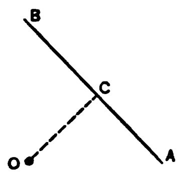

FIG. 73.

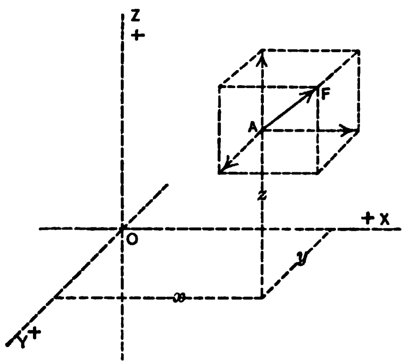

FIG. 74.

moments, whose trace is O (Fig. 73), and let AB be the projection of any force not parallel to the plane. It is evident that, if the force be resolved into components, as stated above, through any point on its line of action, the moment arm of the component in the plane perpendicular to the axis will be OC.

Frequently, it is more convenient to resolve the force into a component, parallel to the axis, and two components, at right angles to each other, in the plane perpendicular to the axis. In this case the algebraic sum of the moments of the last two components will equal the moment of the perpendicular component referred to above (Art. 37).

Thus, let AF represent the line of action of any force F, and x, y, z the coördinates of any point A, on its line of action, with respect to the axes OX, OY and OZ (Fig. 74). To determine the

moment of F with respect to the axis OX, resolve it into components $F \cos \alpha$, $F \cos \beta$ and $F \cos \gamma$, parallel respectively to OX, OY and OZ; α, β and γ being the angles between the line of action of F and the respective axes.

Then the moment of F about OX will be

$$M_x = yF \cos \gamma - zF \cos \beta,$$

and the moments about OY and OZ will be

$$M_y = zF \cos \alpha - xF \cos \gamma,$$
$$M_z = xF \cos \beta - yF \cos \alpha.$$

In determining the sign of the moment, in each case, we determine the direction of rotation by looking along the axis towards the origin from the positive end, calling right-handed rotation plus and left-handed rotation minus.

Hence to determine the moment of a force with respect to any axis, resolve the force into components, passing through any point on its line of action, parallel to and in a plane perpendicular to the axis and compute the algebraic sum of the moments of the components, in the perpendicular plane, about the axis. To determine the sign of the moment of a component about either axis, call the moment plus if its tendency is to rotate right-handed, when looked at from the positive end of the axis towards the origin, and minus, if opposite.

62. A Condition of Equilibrium for any System of Forces Acting through a Point. — Any system of forces acting at a point may be resolved into components along three rectangular coördinate axes (Art. 58), one of which, such as OZ, is parallel to an axis of moments. If the forces are in equilibrium, $\Sigma X = 0$, $\Sigma Y = 0$ and $\Sigma Z = 0$ (Art. 59). Since the moment arms of all the components along a given axis are equal, the sum of the moments of the components about the axis of moments will be equal to zero. Hence, it follows: *When any system of forces acting through a point is in equilibrium, the algebraic sum of their moments about any axis is equal to zero.*

63. Equilibrium of Two Couples in Parallel Planes. — *Two couples acting in parallel planes will balance each other, provided they have equal moments and tend to produce rotation in opposite directions.*

Substitute for the given couples two equivalent couples having equal arms, placed so that the arms are in the same plane, *abcd* (Fig. 75).

The forces of the equivalent couples will all be equal and the resultant of one pair, acting at the middle point of the diagonal *ac* of the parallelogram, will be equal and opposite to the resultant of the other pair, acting at the middle point of the diagonal *bd*. Therefore, the couples balance each other.

Since any couple in the same plane and having an equal and opposite moment will balance the couple $F \times ab$ (Art. 44), it is evident that *two couples will have the same effect, when they have the same moment and tend to turn in the same direction and are situated in the same or parallel planes.*

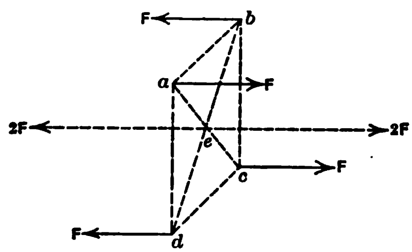

Fig. 75.

64. Characteristics of a Couple. — From the discussion of couples it is evident that a couple is a vector quantity, being determined by its magnitude, direction of rotation and position.

Since its moment is the same about all axes perpendicular to its plane, a couple may be represented by a vector, perpendicular to its plane at any point, whose length represents the magnitude of the couple and whose direction is so taken that the couple tends to turn with right-handed rotation when viewed in the direction indicated. Such a vector is sometimes called the *moment axis of the couple* and may be conveniently used in the discussion of the effects of systems of couples. It is evident that it is necessary to indicate the direction only of the plane of a couple, and not its position, since all couples with equal moments, tending to turn the same way, and situated in parallel planes will have the same effect (Art. 63).

65. Composition of Couples in Parallel Planes. — Since an equivalent system of couples in the same plane can be substituted for any system in parallel planes (Art. 63), the resultant of any system of couples in two, or more, parallel planes will be a couple

whose moment is equal to the algebraic sum of the moments of the couples (Art. 45).

66. Composition of Two Couples in Planes Inclined to Each Other. — Let θ be the angle between the planes of the couples which are perpendicular to the plane of the paper. Lay off the moment axes M_1 and M_2, perpendicular to these planes, to represent the couples (Fig. 76).

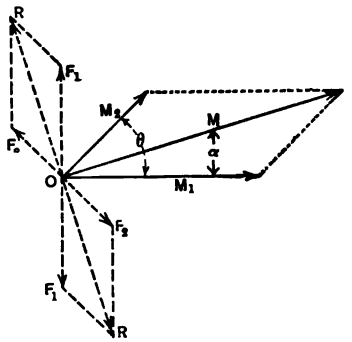

FIG. 76.

The moment axis of the resultant couple will be the vector sum, M, of the moment axes of the couples, M_1 and M_2; for, if we substitute for the given couples two couples with equal arms, placed so that the arms coincide with the intersection of the planes at O, the forces, F_1 and F_2, of the couples will be proportional to the moment axes, M_1 and M_2, and hence their resultant, R, will be proportional to the moment axis, M, of the resultant couple, and, since F_1 and F_2 are perpendicular to M_1 and M_2, the force R will be perpendicular to M. Therefore the magnitude of the resultant couple will be

$$M = \sqrt{(M_1)^2 + (M_2)^2 + 2\,M_1M_2 \cos\theta},$$

and the direction of its moment axis can be found from the equation

$$\sin\alpha = \frac{M_2 \sin\theta}{M},$$

where $\alpha =$ the angle between M and M_1.

When $\theta = 90°$,

$$M = \sqrt{(M_1)^2 + (M_2)^2},$$

and $\sin\alpha = \dfrac{M_2}{M}$; $\cos\alpha = \dfrac{M_1}{M}$.

Conversely, a couple may be resolved into component couples in any two planes which are perpendicular to the same plane as the plane of the couple.

67. Composition of any System of Couples in Planes which are all Perpendicular to a Common Plane. — By the same method of reasoning as was used in connection with the polygon of forces (Art. 32), we can show that the resultant of any system of couples,

situated in planes perpendicular to a common plane, is a couple whose moment axis is the vector sum of the moment axes of the couples; that is, if we represent the moment axes by the sides of a polygon, taken in order, the moment axis of the resultant couple will be represented by the closing side, taken in the opposite order. It follows, that if such a system of couples is in equilibrium, the vector sum of their moment axes is equal to zero.

68. Composition of Any System of Couples. — It is evident (Art. 64) that the moment axes, representing any system of couples in different planes, may be taken to pass through any convenient point. Let the vectors M_1, M_2, M_3 (Fig. 77) represent any such

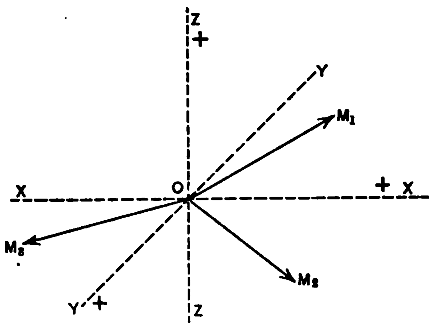

FIG. 77.

system. Since we may resolve each couple into two components in planes at right angles to each other (Art. 66), the same method of reasoning as was used in dealing with forces (Art. 57) will show that each couple may be resolved into three components, with moment axes along OX, OY and OZ. By taking the algebraic sum of each set of components the system may be reduced to three component couples ΣM_x, ΣM_y and ΣM_z, tending to turn about OX, OY and OZ, respectively.

Then, by a method similar to that in Art. 58, we obtain for the magnitude of the resultant couple

$$M = \sqrt{(\Sigma M_x)^2 + (\Sigma M_y)^2 + (\Sigma M_z)^2},$$

and if we let α_m, β_m, γ_m represent the angles which the moment axis of the resultant couple makes with axes OX, OY and OZ, respectively, we have

$$\cos \alpha_m = \frac{\Sigma M_x}{M}, \quad \cos \beta_m = \frac{\Sigma M_y}{M}, \quad \cos \gamma_m = \frac{\Sigma M_z}{M},$$

the quadrant, in which the resultant moment axis is situated, being determined in the same way as in the case of the resultant of a system of forces.

The magnitude and direction of the resultant moment axis could also be determined by constructing a gauche polygon, in the same way as for a system of forces (Art. 58).

If the couples form a balanced system, the vector sums of their components, ΣM_x, ΣM_y and ΣM_z, along each of the three coördinate axes will evidently equal zero.

The quantities ΣM_x, ΣM_y and ΣM_z may be called the moments of the resultant couple, M, about the X, Y and Z axes, respectively.

69. Composition of Parallel Forces, not Confined to the Same Plane. — Refer the system to three coördinate axes, so that

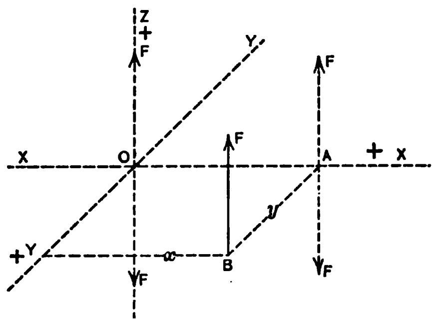

Fig. 78.

the forces are parallel to the axis OZ (Fig. 78). Let F, acting through the point B, whose coördinates are (x, y), be one of the forces. By applying two equal and opposite forces, equal to F, at the points A and O, we may resolve the force F into a force acting along OZ and two couples, $M_x = Fy$ and $M_y = -Fx$, in planes perpendicular to OX and OY, respectively. By resolving each force in the same manner and taking the algebraic sums of the components, we may reduce the system to a single force ΣF, acting along OZ, and two couples, ΣM_x and ΣM_y, tending to turn about OX and OY. In determining the signs of the couples the rule given in Art. 61 should be followed.

Three cases now arise:

Case I. — When $\Sigma F >< 0$, $\Sigma M_x >< 0$, $\Sigma M_y >< 0$.

In this case the force ΣF may be combined with the couple ΣM_x,

giving as the resultant a force ΣF parallel to OZ, acting through a point on OY whose distance from O is equal to

$$y_r = \frac{\Sigma M_x}{\Sigma F}.$$

This force may in turn be combined with the couple ΣM_y, giving for the resultant R, of the entire system, a force whose magnitude is equal to ΣF and perpendicular distance from OY is

$$x_r = \frac{\Sigma M_y}{\Sigma F}.$$

Hence we may state the following rules:

To determine the magnitude of the resultant, find the algebraic sum of the forces.

To determine the line of action of the resultant, find the moments of all the forces in the system about two coördinate axes, OX and OY: then the algebraic sum of the moments about either axis, divided by the resultant, will give the distance of the line of action from that axis.

The position of the resultant must be such that its moment about either axis will be of the same sign as the sum of the moments of the forces about that axis.

When the moment of one of the component couples is equal to zero, it is evident that the axes have been chosen so that the line of action of the resultant force is in the X or the Y plane; and, when the moments of both couples are equal to zero, the line of action of the resultant coincides with OZ.

CASE II. — When $\Sigma F = 0, \quad \Sigma M_x >< 0, \quad \Sigma M_y >< 0.$

In this case the resultant of the system will be a couple whose moment will be equal to

$$M = \sqrt{(\Sigma M_x)^2 + (\Sigma M_y)^2} \text{ (Art. 66)},$$

and the angle θ, which the resultant moment axis makes with OX, may be found from the equations,

$$\cos\theta = \frac{\Sigma M_x}{M}, \quad \sin\theta = \frac{\Sigma M_y}{M}.$$

CASE III. — When $\Sigma F = 0, \quad \Sigma M_x = 0, \quad \Sigma M_y = 0.$

In this case the system of forces will be in equilibrium, and the conditions may be stated as follows:

The algebraic sum of the forces is equal to zero.

The algebraic sums of the moments of the forces about any two axes at right angles to each other in a plane perpendicular to the forces are equal to zero.

70. Conditions of Equilibrium of any System of Parallel Forces. — Since, for any balanced system of parallel forces, the sum of the moments about any two axes at right angles to each other in a plane perpendicular to the forces will be equal to zero, we may state the conditions of equilibrium as follows:

If any system of parallel forces is in equilibrium, the algebraic sum of the forces is equal to zero, and the algebraic sum of the moments of the forces about any axis, in a plane perpendicular to the forces, is equal to zero.

It is evident that three unknown elements in such a system may be determined by the use of these conditions.

71. Resultant of any System of Forces Acting at Different Points and not Confined to the Same Plane. — Let F, F_1, F_2,

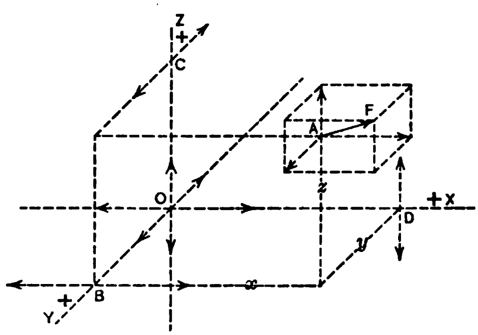

FIG. 79a.

etc., be any system of forces, acting at different points and not confined to the same plane. Refer the system of forces to three coördinate axes, OX, OY and OZ. Let AF be the line of action of one of the forces, F, making the angles α, β and γ with lines through its point of application A, parallel to the axes OX, OY and OZ, respectively; the coördinates of A being (x, y, z) (Fig. 79a).

Resolve F into components, $F \cos \alpha$, $F \cos \beta$, $F \cos \gamma$, parallel respectively to OX, OY, OZ (Art. 57).

Resolve each of these components into an equal and parallel force, acting at O, and two couples; by applying at B and O two forces, equal and opposite and parallel to $F \cos \alpha$; at C and O two

forces, equal and opposite and parallel to $F \cos \beta$; and at D and O two forces, equal and opposite and parallel to $F \cos \gamma$.

In this way, the force F is resolved into three component forces, $F \cos \alpha$, $F \cos \beta$ and $F \cos \gamma$, acting at O along the axes OX, OY and OZ, respectively, and three pairs of component couples,

$$\begin{aligned} M_x &= yF \cos \gamma - zF \cos \beta, \\ M_y &= zF \cos \alpha - xF \cos \gamma, \\ M_z &= xF \cos \beta - yF \cos \alpha, \end{aligned}$$

tending to turn about the axes OX, OY and OZ, respectively. We will adopt the usual signs for designating the directions of the component forces (Art. 58), and determine the signs of the component couples by the rule given in Art. 61.

It should be noted that the sums of the moments of the pairs of couples are equal to the moments of the force F about the three coördinate axes (Art. 61).

Hence, by treating each of the forces in the same way, we may resolve the system into a series of component forces, acting along OX, OY and OZ, and a system of couples, tending to turn about OX, OY and OZ. Then we may add together the component forces, letting ΣX, ΣY and ΣZ equal the algebraic sums of the components along OX, OY and OZ, respectively, and the component couples, letting ΣM_x, ΣM_y and ΣM_z equal the algebraic sums of the components tending to turn about OX, OY and OZ, respectively.

The resultant of the component forces acting at O will be

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}$$

and $$\cos \alpha_r = \frac{\Sigma X}{R}, \quad \cos \beta_r = \frac{\Sigma Y}{R}, \quad \cos \gamma_r = \frac{\Sigma Z}{R} \quad \text{(Art. 58).}$$

The resultant of the component couples will be

$$M = \sqrt{(\Sigma M_x)^2 + (\Sigma M_y)^2 + (\Sigma M_z)^2}$$

and $$\cos \alpha_m = \frac{\Sigma M_x}{M}, \quad \cos \beta_m = \frac{\Sigma M_y}{M}, \quad \cos \gamma_m = \frac{\Sigma M_z}{M} \quad \text{(Art. 68).}$$

In this way, any system of forces may be reduced to a single force R, acting through any point O; and a single couple M, whose moment axis may be represented by a vector through O. To avoid confusion, R and M are represented on a separate sketch (Fig. 79b). Three cases will now be considered.

CASE I. — When $R > 0$ and $M > 0$.

Let θ be the angle between the line of action of R and the moment axis M (Fig. 79b).

If $\theta >< 90°$, resolve the couple M into two components, one component, M_1, with its moment axis coinciding with R, and the other, M_2, with its moment axis at right angles to R. The

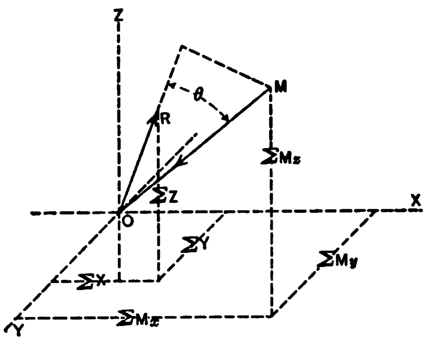

FIG. 79b.

couple M_2 may be combined with R, giving for the resultant a force, equal and parallel to R, whose perpendicular distance from the plane containing R and axis M_2 will be $a_r = \frac{M_2}{R}$ (Art. 46).

Therefore, in the general case, the resultant of any system of forces will be a single force R, at a perpendicular distance from the origin equal to $\frac{M_2}{R}$, and a couple M_1, situated in any plane perpendicular to R.

If $\theta = 90°$, the component couple $M_1 = 0$ and hence $M_2 = M$, and the resultant of the system is a single force equal and parallel to R and $a_r = \frac{M}{R}$.

If $\theta = 0°$, the component couple $M_2 = 0$ and hence $M_1 = M$, and the resultant is a force R, passing through the origin, and a couple M, in any plane perpendicular to R.

If the resultant couple $M = 0$, it is evident that the resultant of the system is the force R, passing through the origin.

CASE II. — When $R = 0$ and $M > 0$.

In this case the resultant of the system is the couple M.

CASE III. — When $R = 0$ and $M = 0$.

In this case, the system of forces is in equilibrium and it follows that the components of R and M along the three coördinate axes must equal zero. Hence

$$\Sigma X = 0, \qquad \Sigma Y = 0, \qquad \Sigma Z = 0,$$
$$\Sigma M_x = 0, \qquad \Sigma M_y = 0, \qquad \Sigma M_z = 0.$$

72. Conditions of Equilibrium of Any System of Forces. — The conditions deduced (Art. 71) may be stated as follows:

If any system of forces, whose lines of action are not in the same plane and do not pass through the same point, is in equilibrium, the algebraic sum of the components of the forces in any three directions at right angles to each other is equal to zero, and the algebraic sum of their moments about any axis is equal to zero.

It is evident that these conditions will enable us to determine six unknown elements in such a balanced system. It may be noted that all the different systems of forces treated in Statics are covered by this proposition and might be treated as special cases under this general head.

73. Problems. — Forces Acting at Different Points and not in the Same Plane.

Problem 1.

Find the resultant of the system of parallel forces shown in Fig. 80.

Solution. — Refer the system to the coördinate axes OX, OY, OZ, with OZ parallel to the forces, the positive ends of the axes being assumed as usual, and the coördinates of the points of intersection of the lines of action of the forces with the Z plane being as indicated.

Then $R = \Sigma F = 80 + 80 - 20 - 40 = 100$ lbs. (acting upward),

$$\Sigma M_x = 80 \times 8 + 20 \times 6 - 80 \times 8 - 40 \times 6 = -120,$$
$$\Sigma M_y = 80 \times 10 + 20 \times 10 - 40 \times 8 - 80 \times 12 = -280 \text{ (Art. 69).}$$

Hence
$$x_r = \frac{280}{100} = 2.8 \text{ ft.},$$
$$y_r = \frac{120}{100} = 1.2 \text{ ft.}$$

In determining the line of action of R, note that its moment about OX must be negative and about OY, negative, when looked at from the positive ends of the axes.

Problem 2.

Find the resultant of the system of parallel forces shown in Fig. 81.

Problem 3.

Find the resultant of the system of parallel forces shown in Fig. 81, if the force of 60 lbs. is replaced by a force of 30 lbs. acting in the same direction.

Problem 4.

A circular table, 9 ft. in diameter, is supported on three legs L, M and N, at equal distances apart (Fig. 82). If a weight of 40 lbs. is placed at A, 30 lbs. at O and 80 lbs. at C, find the stress in each leg. The diameter AC is parallel to LM.

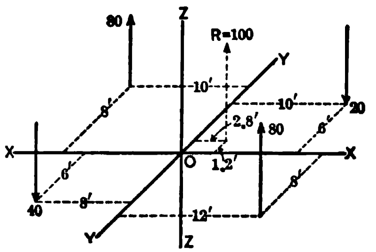

FIG. 80.

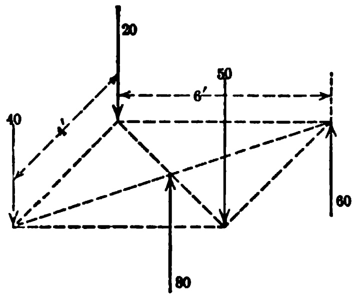

FIG. 81.

Problem 5.

Given a balanced system of parallel forces, acting at the center and vertices of a regular hexagon, and perpendicular to its plane (Fig. 83). Find the unknown forces F, F_1 and F_2, assuming that the forces 80 lbs. and 40 lbs. act downward and the forces 20 lbs. and 60 lbs. act upward. The sides of the hexagon are 6 ft. long.

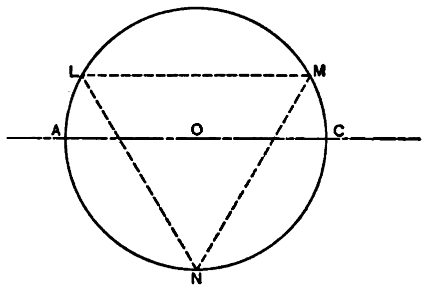

FIG. 82.

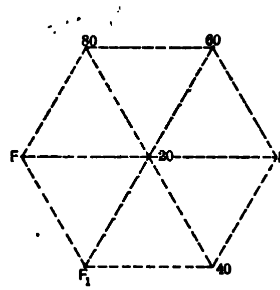

FIG. 83.

Problem 6.

Find the resultant of the system of forces shown in Fig. 84; the force of 60 lbs. acting in the Y plane, at an angle of 30° with the axis OX, the force of 40 lbs. acting in the Z plane, at an angle of 45° with OX, and the force of 80 lbs. acting in a plane parallel to the X plane, and at an angle of 60° with the Z plane.

Solution. — Assume the positive directions along the axes as indicated.

Resolve the forces into components parallel to the coördinate axes and determine the algebraic sums of the components;

$$\Sigma X = 60 \cos 30° + 40 \cos 45° = 80.2,$$
$$\Sigma Y = 40 \cos 45° - 80 \cos 60° = -11.7,$$
$$\Sigma Z = 60 \sin 30° + 80 \sin 60° = 99.3.$$

Then $$R = \sqrt{(80.2)^2 + (11.7)^2 + (99.3)^2} = 128,$$

and $$\cos \gamma_r = \frac{99.3}{128}; \quad \gamma_r = 39° 7' \text{ (Fig. 84).}$$

We will now determine the angle ϕ_r, which the plane containing R and the axis OZ makes with OX.

$$\sqrt{(80.2)^2 + (11.7)^2} = 81.0.$$

Hence $$\sin \phi_r = \frac{11.7}{81.0} \quad \text{and} \quad \phi_r = 8° 18',$$

and the intersection of this plane with the Z plane will be OA (Fig. 84).

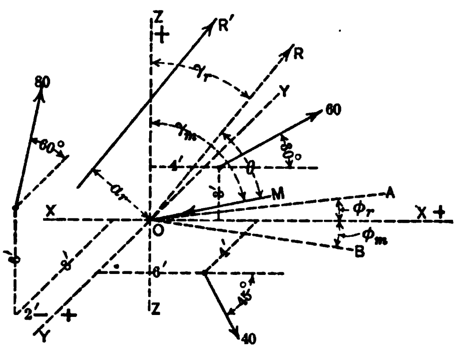

Fig. 84.

Compute the moments of the forces about OX, OY and OZ;

$$\Sigma M_x = 6 \times 80 \cos 60° + 8 \times 80 \sin 60° = 794,$$
$$\Sigma M_y = 2 \times 80 \sin 60° + 3 \times 60 \cos 30° - 4 \times 60 \sin 30° = 174,$$
$$\Sigma M_z = 2 \times 80 \cos 60° + 6 \times 40 \sin 45° - 4 \times 40 \cos 45° = 137.$$

Then $$M = \sqrt{(794)^2 + (174)^2 + (137)^2} = 824,$$

and $$\cos \gamma_m = \frac{137}{824}; \quad \gamma_m = 80° 34' \text{ (Fig. 84).}$$

Since the moment axis, M, does not coincide with and is not at right angles to R, we will resolve M into two components, one with the axis coinciding with R, and one at right angles to R, as follows:

Find the angle ϕ_m, which the plane containing OZ and M makes with OX.

$$\sqrt{(794)^2 + (174)^2} = 813.$$

Hence $$\sin \phi_m = \frac{174}{813} \quad \text{and} \quad \phi_m = 12° 22',$$

and the intersection of this plane with the Z plane will be OB (Fig. 84).

Hence the angle between the planes AOZ and BOZ will be equal to

$$\phi_r + \phi_m = 20° 40'.$$

We can find the angle θ, between M and R, by means of the formula from spherical trigonometry, giving the relation between the face and diedral angles of the triedral angle,

$$\cos\theta = \cos\gamma_m \cos\gamma_r + \sin\gamma_m \sin\gamma_r \cos AOB.$$

Substituting the values for γ_m, γ_r and AOB we obtain

$$\cos\theta = 0.711 \quad \text{and} \quad \theta = 44^\circ 41'.$$

Resolving the couple M into a component M_1, whose moment axis coincides with R and a component M_2, at right angles to M_1 we have

$$M_1 = M\cos\theta = 586,$$
$$M_2 = M\sin\theta = 579.$$

Combining M_2 with R, we have for the resultant of the system a force equal and parallel to R,

$$R' = 128 \text{ lbs.},$$

whose distance from R is equal to

$$a_r = \frac{M_2}{R} = \frac{579}{128} = 4.52 \text{ ft.},$$

measured from O in a direction perpendicular to the plane containing R and M, as shown in Fig. 84, and a couple, whose moment is

$$M_1 = 586 \text{ ft.-lbs.},$$

in a plane perpendicular to R', tending to turn right-handed when looked at in the downward direction, along the line of action of R'.

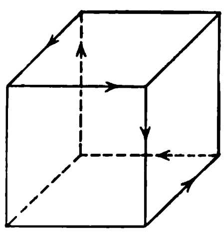

FIG. 85.

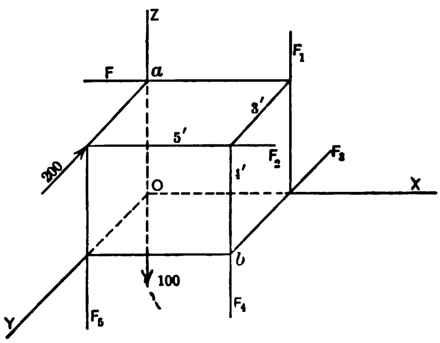

FIG. 86.

Problem 7.

The lines of action of a system of equal forces coincide with the edges of a cube (Fig. 85). Find their resultant. The length of the side of the cube $= a$; the magnitude of one of the forces $= F$.

Problem 8.

The forces 100 lbs., 200 lbs., F, F_1, F_2, F_3, F_4, F_5, acting at the corners of a rectangular prism, and parallel to the sides as shown (Fig. 86), are in equilibrium. Find the unknown forces.

Solution. — Assume the coördinate axes OX, OY and OZ, as indicated, and

assume that the unknown forces all act in the positive directions along the axes, to which they are respectively parallel.

Then
$$\begin{aligned}
\Sigma X &= F + F_2 = 0,\\
\Sigma Y &= -200 + F_3 = 0,\\
\Sigma Z &= -100 + F_1 + F_4 + F_5 = 0,\\
\Sigma M_x &= 200 \times 4 + 3F_5 + 3F_4 = 0,\\
\Sigma M_y &= 4F + 4F_2 - 5F_1 - 5F_4 = 0,\\
\Sigma M_z &= -3F_2 + 5F_3 = 0.
\end{aligned}$$

Solving these equations we obtain:

$$\begin{aligned}
F_3 &= 200 \text{ lbs.},\\
F_2 &= 333 \text{ lbs.},\\
F &= -333 \text{ lbs.},\\
F_5 &= 100 \text{ lbs.},\\
F_4 &= -367 \text{ lbs.},\\
F_1 &= 367 \text{ lbs.},
\end{aligned}$$

showing that the forces F and F_4 act in the opposite directions to those assumed.

Problem 9.

Solve Problem 8, assuming that the force F_4 acts along the diagonal Ob of its face (Fig. 86).

Problem 10.

The frame (Fig. 87), with face ABC horizontal, is acted upon at O by a vertical force of 400 lbs. and a force of 500 lbs., parallel to the side BC, and is supported by a pin joint at A, a vertical force at C, a vertical force at B and a horizontal force at B, acting in a plane perpendicular to AB. Find the components of the unknown forces acting at A, B and C, each member of the frame being 10 ft. long.

Problem 11.

Find the stresses in the members of the frame given in Problem 10 (Fig. 87).

Problem 12.

The frame (Fig. 88) has the form of a right triangular prism, with horizontal members $ab = ac = bc = de = ef = df = 6$ ft. and vertical members $ad = be = cf = 8$ ft. The frame is subjected to the vertical forces 400 lbs., 800 lbs. and 600 lbs., the horizontal force of 500 lbs., perpendicular to bc, and the horizontal force of 300 lbs., along ac, acting in the directions indicated; and is supported at d, e and f by the vertical forces F, F_1 and F_2, and by three horizontal forces, F_3, F_4 and F_5, respectively perpendicular to de, ef and fd. Find the supporting forces.

Solution. — In the following solution, the details in regard to the resolution of forces and computation of distances are omitted. The student should supply these from the data given. As in the solution of problems involving systems of forces in the same plane, it is not necessary to complete the entire solution by referring the forces to one set of coördinate axes, but different sets of axes may be chosen to simplify the solution of the problem.

First, we will take an axis of moments coinciding with *df*.

Then $\Sigma M = -800 \times 5.20 - 500 \times 0.5 \times 8 + F_1 \times 5.20 = 0,$

and $F_1 = 1184$ (acting upward).

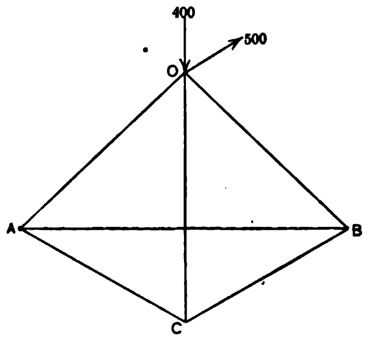

FIG. 87.

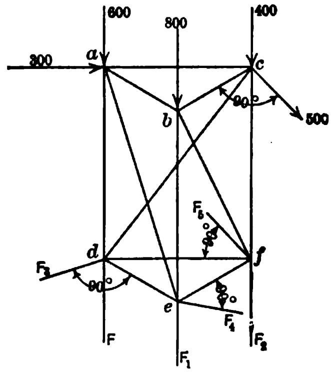

FIG. 88.

Take an axis of moments coinciding with *ed*.

Then $\Sigma M = 300 \times 0.866 \times 8 + 400 \times 5.20 + 500 \times 0.5 \times 8 - 5.20\, F_2 = 0,$

and $F_2 = 1186$ (acting upward).

Adding the vertical components we have

$$-600 - 800 - 400 + 1184 + 1186 - F = 0,$$

and $F = 570$ (acting downward).

Take a vertical axis of moments through the intersection of F_4 and F_5.

Then $\Sigma M = 500 \times 6 + 300 \times 6.93 - F_3 \times 0.866 \times 6.93 - F_3 \times 0.5 \times 6 = 0,$

and

$F_3 = 564$ (acting toward the left).

Take an axis of moments coinciding with *cf*.

Then $\Sigma M = -564 \times 0.5 \times 6 + 6\, F_4 = 0,$

and $F_4 = 282$ (acting toward the left).

Take an axis of moments coinciding with *ad*.

Then $\Sigma M = 500 \times 0.5 \times 6 + 282 \times 0.5 \times 6 - 6\, F_5 = 0,$

and $F_5 = 391$ (acting toward the back).

Problem 13.

Find the stresses in the members of the frame given in Problem 12 (Fig. 88).

Solution. — Denote the stresses as follows:

Stress in *ab* $= F_6$,	Stress in *ae* $= F_{12}$,
Stress in *bc* $= F_7$,	Stress in *cd* $= F_{13}$,
Stress in *ac* $= F_8$,	Stress in *bf* $= F_{14}$,
Stress in *ad* $= F_9$,	Stress in *de* $= F_{15}$,
Stress in *be* $= F_{10}$,	Stress in *df* $= F_{16}$,
Stress in *cf* $= F_{11}$,	Stress in *ef* $= F_{17}$.

For convenience these forces, as well as the magnitudes and directions of the supporting forces, found in Problem 12, should be indicated on a sketch of the frame. The conditions of equilibrium (Art. 59) will apply to the

systems of forces acting at each joint of the frame. An inspection of the frame, however, shows that there are four unknown forces acting at every joint, and therefore a solution cannot be made by applying the conditions of equilibrium to the forces acting at any one joint alone. But, since the forces exerted by each member on the two joints it connects are the same, it will be found that, when the conditions of equilibrium are applied to the forces acting at the six joints of the frame, we have conditions enough to enable us to determine all the unknown forces.

As in Problem 12, it is not necessary to resolve the forces acting at the different joints into components in three fixed directions, but the directions of the axes may be changed to simplify the solution. In the following solution, the directions assumed for the X axis only will be indicated in each case, the Z axis being always vertical, and the Y axis perpendicular to Z and X. The directions of the unknown forces are assumed, and the kind of stress determined in the usual way.

Joint b: X axis along bc.

$$\Sigma Y = F_6 = 0.$$

Joint a: X axis along ab.

$$\Sigma Y = 300 \cos 30° - F_8 \cos 30° = 0,$$
$$F_8 = 300 \text{ (Compression)},$$
$$\Sigma X = F_6 - 0.6 F_{12} + 300 \sin 30° - F_8 \sin 30° = 0,$$
$$F_{12} = 0,$$
$$\Sigma Z = -600 - 0.8 F_{12} + F_9 = 0,$$
$$F_9 = 600 \text{ (Compression)}.$$

Joint c: X axis along ac.

$$\Sigma Y = 500 \sin 30° - F_7 \sin 60° = 0,$$
$$F_7 = 289 \text{ (Compression)},$$
$$\Sigma X = 500 \cos 30° + F_7 \cos 60° + F_8 - 0.6 F_{13} = 0,$$
$$F_{13} = 1462 \text{ (Tension)},$$
$$\Sigma Z = -400 - 1462 \times 0.8 + F_{11} = 0,$$
$$F_{11} = 1570 \text{ (Compression)}.$$

Joint b: X axis along bc.

Since $F_6 = 0$,

$$\Sigma X = F_7 - 0.6 F_{14} = 0,$$
$$F_{14} = 481 \text{ (Tension)}.$$

Joint e: X axis along ef.

Since $F_{12} = 0$,

$$\Sigma Z = 1184 - F_{10} = 0,$$
$$F_{10} = 1184 \text{ (Compression)},$$
$$\Sigma Y = 282 - F_{15} \cos 30° = 0,$$
$$F_{15} = 325 \text{ (Compression)},$$
$$\Sigma X = F_{15} \sin 30° - F_{17} = 0,$$
$$F_{17} = 162 \text{ (Tension)}.$$

Joint f: X axis along ef.

$$\Sigma Y = -391 \cos 60° + F_{16} \cos 30° = 0,$$
$$F_{16} = 226 \text{ (Compression)}.$$

§ 4. Distributed Forces.

74. Types of Distributed Forces. — As stated in Art. 20, the place of application of a force is always a surface or a volume.

In the propositions taken up thus far, we have, for convenience, considered different systems of forces as acting at, or through, points. It will appear presently that these propositions will apply to the resultants of forces, which are distributed over surfaces or through volumes; and that in the solution of problems, where we have considered forces, which were distributed through volumes or over surfaces, as acting at points, we have really been dealing with the resultants of those forces.

In certain cases, it is convenient to assume that a force is distributed along a line, in the same way that we assume that a force is concentrated at a point, it being evidently impossible for a force to have for its place of application a single line. The force exerted by gravity on a slender rod, or the force exerted on a very narrow surface could be treated in this way. If the rod, or the surface, were divided up into very small lengths, the vectors representing the resultant forces, acting on these elementary lengths, would all pass through the same line.

We will now take up the discussion of the manner in which the propositions in regard to concentrated forces may be applied to distributed forces. In the cases to be considered, the elements of the distributed forces will be parallel to each other, and hence the forces may be treated as being made up of a very large number of very small parallel forces.

Three kinds, or types, of distributed forces will be considered, namely:

(1) Force distributed along a line.
(2) Force distributed over an area.
(3) Force distributed through a volume.

In discussing type (1), we shall confine ourselves to the straight line and in discussing type (2), to the plane area.

75. Intensity of a Distributed Force. — When a force is distributed over an area, its intensity at any point will be the number of units of force per unit of area at that point. If the force is uniformly distributed, the intensity, p, will be equal to the force, P, acting on the entire area, divided by the area, A, or

$$p = \frac{P}{A}.$$

If the force is not uniformly distributed, the intensity at any point will be equal to

$$p = \frac{dP}{dA},$$

where $\frac{dP}{dA}$ is the limit of the ratio of the force, acting on a very small element of the area, to that element as it approaches zero as a limit.

In a similar manner if a force, P, is distributed through a volume, V, its intensity at any point will be the number of units of force acting per unit of volume at that point. If uniformly distributed, the intensity will be equal to

$$p = \frac{P}{V},$$

and if not uniformly distributed,

$$p = \frac{dP}{dV}.$$

In case a force is assumed to be distributed along a line, the intensity at any point will be the number of units of force acting per unit of length at that point. If uniformly distributed, the intensity will be equal to

$$p = \frac{P}{L},$$

and if not uniformly distributed,

$$p = \frac{dP}{dL}.$$

The unit of intensity will be a unit force acting through unit volume, or on a unit surface, or along a unit length, as, for example, the pound per cubic inch, the pound per square inch, the pound per inch, etc.

76. Resultant of a Distributed Force. — Case I. Force Distributed along a Straight Line.

Assume that the force is perpendicular to the line AB (Fig. 89) and is distributed along the line in such a manner that its intensity p, at any point e, is represented by the ordinate ef, between AB and the line Cd. Assume the axes of X and Y as indicated. Then the force acting at the point e, on a length dx of the line, will be pdx, and the entire distributed force will consist of a very

large number of such elementary forces which will be parallel to each other and situated in the same plane.

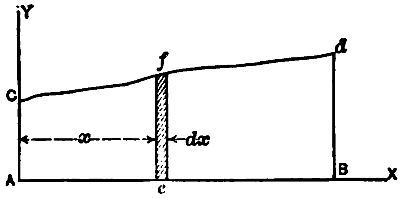

FIG. 89.

The resultant can therefore be determined by the method in Art. 48.

Hence $$R = \Sigma F = \int p\,dx,$$

and, if x_r equals the distance of the line of action of the resultant from A, its moment about A will be

$$Rx_r = \Sigma Fx = \int px\,dx,$$

and $$x_r = \frac{\int px\,dx}{\int p\,dx}.$$

If $p = \phi\,(x)$, R and x can be found by integration; if not, a solution can be made by dividing the line into small finite lengths

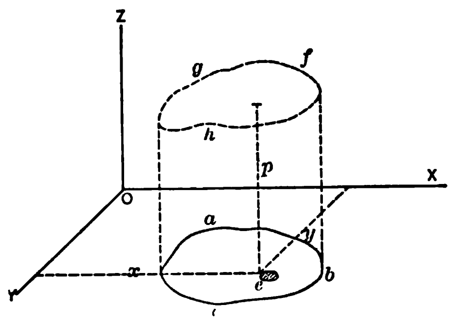

FIG. 90.

and finding the resultant of the parallel forces acting on these lengths by the method of Art. 48. Such a solution will be approximate, but can always be made exact enough for the solution of

engineering problems. It is evident that, if plotted to the right scale, the area $ACdB$ would equal the magnitude of the resultant.

CASE II. Force Distributed over a Plane Surface.

Assume that the force is perpendicular to the area abc and is distributed over the surface in such a manner that its intensity p, at any point e, is represented by the length of the perpendicular between the surface abc and the surface fgh (Fig. 90).

Refer the surface to the coördinate axes OX and OY, in the plane of the area, and let the coördinates of the point e be (x, y). The force acting on an elementary area dA, at the point e, will be $p\,dA$ and the entire distributed force will consist of a very large number of such elementary forces, parallel to each other, but not in the same plane.

The resultant can therefore be determined by the method in Art. 69.

Hence
$$R = \Sigma F = \int p\,dA,$$

and, if x_r and y_r are the coördinates of its line of action, the moment of the resultant about OY will be

$$Rx_r = \int px\,dA,$$

and about OX,

$$Ry_r = \int py\,dA,$$

and
$$x_r = \frac{\int px\,dA}{\int p\,dA}, \qquad y_r = \frac{\int py\,dA}{\int p\,dA}.$$

If $p = \phi\,(x, y)$, R, x_r and y_r can be found by integration; if not, an approximate solution can be made by dividing the area into small finite areas and finding the resultant of the parallel forces acting on these areas by the method of Art. 69. It is evident that if the prism $abcfgh$ is plotted to the right scale, its volume will be equal to the magnitude of R.

Case III. Force Distributed through a Volume.

Refer the volume to three coördinate axes, OX, OY, OZ, and let the force acting be parallel to OZ. If we let p equal the intensity of the force at any point in the volume whose coördinates are (x, y, z), the force acting on an elementary volume dV, at this

point, will be $p\,dV$. Considering the entire volume, we have a system of elementary forces, similar to that in Case II, and by the same method of analysis we have $R = \int p\,dV,$

$$x_r = \frac{\int px\,dV}{\int p\,dV}, \quad y_r = \frac{\int py\,dV}{\int p\,dV},$$

x_r and y_r being the coördinates of any point on the line of action of the resultant, R.

If $p = \phi\,(x, y, z)$, R, x_r and y_r can be found by integration; if not, an approximate method similar to that stated under Case II can be employed.

As in the case of the line and area, the force could evidently be represented graphically.

77. Stress. — In Art. 40, the stress in a straight rod was defined as the force exerted at any cross section by the part of the rod on one side of the section on that on the other side.

A broader definition of the term will be the following: A *stress* is a force exerted at a surface of contact of two contiguous bodies or at a section between two parts of the same body. While the term is sometimes used in other ways in Mechanics it is customary in Engineering to restrict it to a force acting on a surface as defined above. Unless otherwise stated, a plane surface is always meant.

For example, if a cylinder of iron weighing 600 lbs. rests in a vertical position on a stone foundation, a stress of 600 lbs. will be exerted on the bearing surface, between the iron and the stone; likewise the stress on a horizontal cross section, half way between the top and bottom of the cylinder, will be equal to 300 lbs.

78. Components of Stress. — In case the stress acting on an element of an area is not perpendicular to the surface, it can be resolved into components perpendicular to and along the surface. The first is called the *normal component* and will be either tension or compression. The second is called the *tangential* or *shearing component*. The resultant of these components is called the *resultant stress* on the element.

In case the shearing components of the stress at different points in the surface are not parallel, they can be resolved into two sets of tangential components at right angles to each other. In this way, any stress may be resolved into three sets of components at

right angles; and its resultant determined by the method of Art. 71.

If a stress has no shearing component, or if it is a shearing stress with no normal component, it is called a *simple stress*. Any other stress may be called a *complex stress*.

All stresses can therefore be resolved into simple stresses which may be summarized as follows:

$$\text{Components of Stress}\left\{\begin{array}{l}\text{Normal}\left\{\begin{array}{l}\text{Tension,}\\ \text{or Compression.}\end{array}\right.\\ \text{Shear}\end{array}\right.$$

Unless otherwise stated, when we speak of the stress on any surface we mean the resultant of the entire stress.

If the direction of a complex stress at every point in a surface is the same, the resultant may be found without resolving the stress into components, the deductions in the following articles being applicable directly to the resultant stresses on the elements, into which the area is divided, as well as to the components of the stresses on those elements.

79. Intensity of Stress. — The intensity of a stress, at any point in a surface, may be defined as the stress per unit of area at that point. Since a stress is a force distributed over an area, the method of determining its intensity is the same as that for the plane surface (Art. 75).

The *units in which the intensity of stress*, or *unit stress*, as it is frequently called, will be expressed are pounds per square inch, pounds per square foot, kilograms per square centimeter, etc. When the intensity of stress is the same at every point we call it a *uniform stress*. When the intensity is not the same at every point we call it a *varying stress*.

80. Center of Stress. — The point through which the resultant of a stress passes is called the center of stress. If p equals the intensity of a simple stress at any point in a plane surface, R, the resultant stress and x_1 and y_1, the coördinates of the center of stress, we have, by adapting the formulas in Art. 76,

$$R = \int p\,dA, \qquad x_1 = \frac{\int px\,dA}{\int p\,dA}, \qquad y_1 = \frac{\int py\,dA}{\int p\,dA}.$$

If $p = \phi(x, y)$, the resultant stress and the coördinates of the center of stress can be found by integration. If not, they can be found approximately by the method stated in Art. 76.

In some cases, p may be positive for certain points in a surface and negative for others. For example, a simple normal stress may be tension over part of a surface and compression over the remainder. In certain cases of this kind the resultant of the stress will be a couple, instead of a single force, in which case

$$R = \int p\,dA = 0,$$

and, if the Y axis is perpendicular to the plane of the couple, the moments of the stress will be

$$\int px\,dA >< 0, \qquad \int py\,dA = 0.$$

81. Uniform Stress. — In this case p = a constant and, if we let A = the area over which the stress is distributed, the formulas (Art. 80) become

$$R = pA, \qquad x_1 = \frac{\int x\,dA}{A}, \qquad y_1 = \frac{\int y\,dA}{A}.$$

82. Uniformly Varying Stress. — A uniformly varying stress is one whose intensity at any point is proportional to the distance of that point from a straight line, in the plane of the area over which the stress is distributed. This line, at every point on which the intensity of stress will be zero, is called the *neutral axis,* or zero line. If we refer the area to the coördinate axes OX and OY, so chosen that OY coincides with the neutral axis, the intensity of stress at any point whose coördinates are (x, y) will be $p = ax$, where a is a constant, which is equal to the intensity at a distance unity from the neutral axis.

Hence for this case the equations (Art. 80) become

$$R = a\int x\,dA, \qquad x_1 = \frac{\int x^2\,dA}{\int x\,dA}, \qquad y_1 = \frac{\int xy\,dA}{\int x\,dA}.$$

A familiar example of this kind of stress is the case of water pressure on a vertical, or inclined, plane surface; the neutral axis being the line of intersection of the surface with the surface of the water.

83. Problems. — Distributed Forces. —

Problem 1.

The force acting along a straight rod, AB, is distributed in such a way that the intensity at any point may be represented by the ordinate to the straight line, AC (Fig. 91). If the intensity at B is equal to 10 lbs. per inch, find the resultant force.

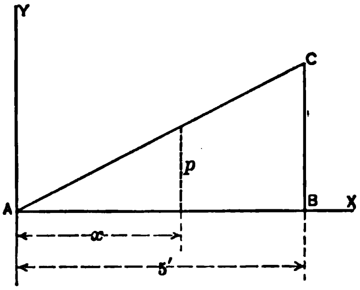

FIG. 91.

Solution. — If we take an origin at A and the X axis coinciding with AB. the intensity, p, at any point in the rod, at a distance x from A will be

$$p = \frac{10}{60}x = \frac{x}{6}.$$

Then $$R = \int p\,dx = \int_0^{60} \frac{x\,dx}{6} = 300 \text{ lbs.} \quad \text{(Art. 76)},$$

and $$Rx_r = \int px\,dx = \int_0^{60} \frac{x^2\,dx}{6} = 12{,}000 \text{ in.-lbs.}$$

Hence $$x_r = 40 \text{ ins.}$$

Problem 2.

Prove that, whenever a force is distributed along a line in the manner indicated in Problem 1, the resultant will be equal to the product of the length of the line and the average intensity of the force; and the distance of the line of action of the resultant, from the point of zero intensity, will be two-thirds of the length of the line.

Such a force is called a *uniformly varying force.*

Problem 3.

A force is distributed along the line AB in the manner indicated (Fig. 92), the intensities in pounds per foot at the points A, C, D, E and B being 100, 100, 300, 0 and 150 as indicated. Find the resultant.

Solution. — The force acting on the part AD may be divided into a uniformly distributed force of 100 lbs. per ft., the resultant of which will be 1000 lbs., acting at a distance of 5 ft. from A: and a uniformly varying force, whose resultant (Prob. 2) will be 600 lbs., acting at a distance of 8 ft. from A; the resultant of the force acting along the portion DE will be 900 lbs., acting at

a distance of 12 ft. from A; and the resultant of the force acting along EB will be 225 lbs., acting at a distance of 18 ft. from A in the upward direction;

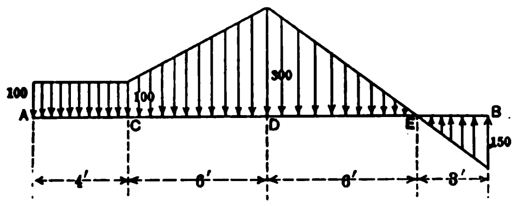

Fig. 92.

all the other forces acting downward. The resultant of the system can now be found by treating these components as a system of parallel forces (Art. 48).

Hence
$$R = \Sigma F = 1000 + 600 + 900 - 225$$
$$= 2275 \text{ lbs. (acting downward)},$$
$$Rx_r = 1000 \times 5 + 600 \times 8 + 900 \times 12 - 225 \times 18$$
$$= 16{,}550 \text{ ft.-lbs.},$$
$$x_r = 7.27 \text{ ft.}$$

Problem 4.

Assuming that the system of forces in Problem 3 is balanced by two parallel forces applied at the points A and E, find the unknown forces.

Problem 5.

A force is distributed along a line AB in such a manner that its intensity at any point, at a distance x from A, is equal to

$$p = 10x - \frac{x^2}{2},$$

and is balanced by a uniformly varying force, acting along AC, and a uniform force, acting along DB. Find the intensities of the balancing forces at the points A, C, D and B (Fig. 93).

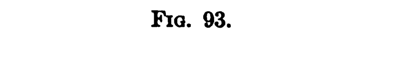

Fig. 93.

Problem 6.

Find the resultant of a force, distributed along a line 4 ft. long, the intensity of which in pounds per foot at any point, whose distance from the end of the line is x, is equal to

$$p = 5 \sin x.$$

Problem 7.

A force is distributed along a line, 10 ft. long, in such a manner that its intensity in pounds per foot, at a distance x from one end of the line, is equal to $p = 80\sqrt{x}$, and is balanced by two parallel forces, acting at the ends of the line. Find the unknown forces.

Problem 8.

The stress on a rectangular surface (Fig. 94) varies uniformly in such a manner that the intensity at any point is equal to $p = 40\,x$ lbs. per sq. ft., x being the distance of the point from the line OY, 4 ft. from and parallel to the side of the rectangle. Find the resultant stress and the coördinates of the center of stress.

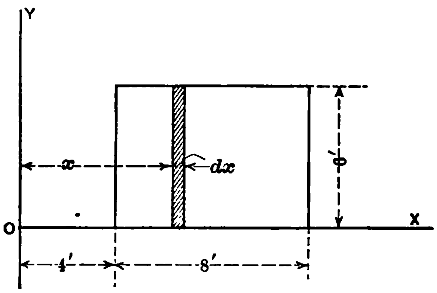

FIG. 94.

Solution. — Refer the area to OY and OX as coördinate axes. In the formulas (Art. 76),

$$R = \int p\,dA, \qquad Rx_r = \int px\,dA,$$

we may substitute for dA the area of a strip of width dx, parallel to OY.

Hence $$dA = 6\,dx.$$

Since the intensity will be the same at every point on the strip,

$$p\,dA = 40\,x\;6\,dx = 240\,x\,dx,$$

$$R = 240\int_4^{12} x\,dx = 15{,}360 \text{ lbs.},$$

$$Rx_r = 240\int_4^{12} x^2\,dx = 133{,}120 \text{ ft.-lbs.},$$

and $$x_r = 8.67 \text{ ft.}$$

Since the resultant stress on each strip will act through its middle point the other coördinate of the center of stress will be

$$y_r = 3 \text{ ft.}$$

Problem 9.

Find the resultant stress on the rectangular surface (Fig. 94), if the intensity at any point is equal to

$$p = 10\,x^{\frac{3}{2}},$$

referring to the same coördinate axes. Also find the coördinates of the center of stress.

Problem 10.

Find the resultant stress on the rectangular surface (Fig. 94), if the intensity of stress at any point, referring to the axes OX and OY, is equal to

$$p = 8\,xy^2;$$

also find the coördinates of the center of stress.

In the solution of this problem we may let $dA = dx\,dy$, whence the formulas (Art. 76) become

$$R = \int\int p\,dx\,dy,$$

$$Rx_r = \int\int px\,dx\,dy,$$

$$Ry_r = \int\int py\,dx\,dy,$$

which can be solved by double integration.

84. Gravity. — In the attraction of the Earth on any mass we have an example of a force distributed through a volume. Without appreciable error we can assume that the forces exerted on the different particles are all parallel to each other. If we let w equal the weight per unit of volume, at any point in a given mass, W, its entire weight, and x_1 and y_1, the coördinates of the line of action of the resultant force exerted by gravity on the mass, the equations (Case III, Art. 76) become

$$W = \int w\,dV, \qquad x_1 = \frac{\int wx\,dV}{\int w\,dV}, \qquad y_1 = \frac{\int wy\,dV}{\int w\,dV}.$$

If the mass is homogeneous, w = constant, and, if we let V = its volume, the equations reduce to

$$W = wV, \qquad x_1 = \frac{\int x\,dV}{V}, \qquad y_1 = \frac{\int y\,dV}{V}.$$

85. Attraction. — Gravitation and Other Forces. — According to the law of inertia, matter in itself has no power to change its state. Matter does act on other matter, however, with forces of attraction, or repulsion, in accordance with certain laws which have been amply verified by experiment.

According to Newton's Law of Universal, or Cosmical, Gravitation, every particle of matter attracts every other such particle with a force, directly proportional to the product of the masses, and inversely proportional to the square of the distance of the particles.

If the masses of the particles are m and m_1 and the distance between them is c, the law may be expressed algebraically as follows:

$$F = K\frac{mm_1}{c^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where F is the attractive force between the particles and K is a constant, determined by experiment; its numerical value depending on the units in which F, m, m_1 and c are expressed. In the application of this law the attraction of one particle on another may be regarded as acting at a point. In interpreting formula (1), however, it must be remembered that, since the particles have finite masses, they must have finite dimensions and hence, as the distance between them is diminished, c cannot be less than a certain finite quantity and the maximum value of F, which will occur when the particles are in contact, will be a finite quantity. If c were assumed to equal zero in any case, F, for finite values of m and m_1, would equal ∞, which would be impossible.

While formula (1) expresses the law of gravitation, the general algebraic expression for the law of attraction would be

$$F = K\frac{mm_1}{\phi(c)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\phi(c)$ is some function of the distance between the particles, depending on the nature of the attractive force, K is a constant, and m and m_1 other quantities than the masses of particles. The following discussion will apply to the law as expressed by formula (1).

It is evident that K is equal to the force with which two particles of unit mass at a unit distance apart attract each other. If we divide equation (1) by m_1 we have

$$\frac{F}{m_1} = a = K\frac{m}{c^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where a is the acceleration which would be produced in the mass m_1 by the attraction of the mass m at a distance c.

The quantity $K\frac{m}{c^2}$ would also equal the force of attraction exerted by the mass m on a mass *unity* at a distance c. For brevity this is called the *attraction at the point*, at which the unit mass is situated, exerted by the mass m.

In determining the *attraction at a point* exerted by any mass the attractions of the separate particles may be resolved into components, parallel to coördinate axes, and the resultants of these components found by the methods given in Art. 76.

The attraction at a point exerted by any mass may also be called the *strength of field*. The last word is an abbreviation for *the field of force*, by which is meant the space through which the attraction of the mass is exerted.

Electrostatic and magnetic attraction and repulsion are other examples of attractive forces, which are governed by laws similar to that of gravitation. Thus, if in formula (1) we let m and m_1 equal the quantities of electricity, or magnetism, contained by two particles at a distance c from each other, the formula would give the force due to the electric, or magnetic, attraction or repulsion exerted between the particles, depending on whether the particles were dissimilarly or similarly electrified, or magnetized.

Hence, by changing the system of units, the formulas deduced by the application of the law of gravitation may be used to determine electric and magnetic attractions wherever the distribution of electricity, or magnetism, is the same as the distribution of mass in the bodies considered; and the same would apply to any other attractive forces governed by similar laws.

86. Problems. — Attraction at a Point. — The solution of all of the following problems is based on the law of attraction of a particle

$$F = K\frac{m}{c^2},$$

the body in every case being assumed to be homogeneous.

Since in each case the elementary forces all pass through the same point, the resultant attraction may be determined by the method of resolution of forces (Art. 58).

Problem 1. — Slender Straight Rod of Uniform Section.

Find the attraction at O of the rod AB (Fig. 95). Let M = the mass of the rod and $x = OD$ be the perpendicular distance from O to the line AB.

Solution. — Assume the axes OX and OY, as indicated, and let α, θ_1 and θ_2 equal the angles between OX and the straight lines OE, OA and OB, respectively. Let m = the mass per unit length of the rod. Then the attraction at O of a particle of length dy, situated at any point E, whose coördinates are (x, y) will be

$$dF = \frac{Km}{(OE)^2}dy = \frac{Km\cos^2\alpha}{x^2}dy. \quad \ldots \ldots \quad (1)$$

Since $$y = x \tan \alpha, \qquad dy = \frac{x\,d\alpha}{\cos^2 \alpha}$$

and $$dF = \frac{Km}{x}\,d\alpha. \qquad (2)$$

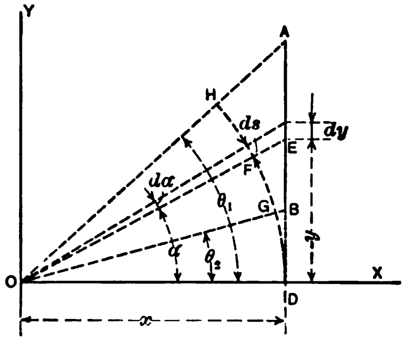

FIG. 95.

Resolve dF into components parallel to OX and OY and treat all the particles of the rod in the same way.

Then $$\Sigma X = \frac{Km}{x}\int_{\theta_2}^{\theta_1} \cos \alpha\, d\alpha,$$

$$\Sigma Y = \frac{Km}{x}\int_{\theta_2}^{\theta_1} \sin \alpha\, d\alpha.$$

Integrating, $$\Sigma X = \frac{Km}{x}(\sin \theta_1 - \sin \theta_2),$$

$$\Sigma Y = \frac{Km}{x}(\cos \theta_2 - \cos \theta_1).$$

Then $$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2} = \frac{Km}{x}\sqrt{2\{1 - \cos(\theta_1 - \theta_2)\}}$$

$$= \frac{2\,Km}{x}\sin\frac{\theta_1 - \theta_2}{2}, \qquad (3)$$

and $$\tan \alpha_r = \frac{\Sigma Y}{\Sigma X} = -\frac{\cos\theta_1 - \cos\theta_2}{\sin\theta_1 - \sin\theta_2} = \tan\frac{\theta_1 + \theta_2}{2},$$

and $$\alpha_r = \frac{\theta_1 + \theta_2}{2}. \qquad (4)$$

Therefore, the line of action of R bisects the angle AOB, subtended by AB at O.

When $x = 0$ and the point O coincides with D, on the axis of the rod produced, equation (3) becomes indeterminate and fails to give a solution.

In this case let $DA = y_1$ and $DB = y_2$.

Then equation (1) becomes

$$dF = Km\frac{dy}{y^2},$$

and $$R = Km\int_{y_2}^{y_1}\frac{dy}{y^2} = Km\left(\frac{1}{y_2} - \frac{1}{y_1}\right)$$

$$= \frac{Km\,(y_1 - y_2)}{y_1 y_2} = \frac{KM}{y_1 y_2}, \qquad (5)$$

where M = the entire mass of the rod.

If the point O is taken at the end of the rod, $y_2 = 0$ and equation (5) gives $R = \infty$. This is impossible, however, and in interpreting the equation it must be remembered that by the attraction at a point we mean the attraction on a unit mass at that point and that it would be impossible to have y_2, the distance between the end of the rod and the center of any finite particle, equal to zero. On the other hand if $y_1 = \infty$,

$$R = \frac{Km}{y_2}. \qquad (6)$$

Hence in this case, R would always be a finite quantity, no matter what the length of the rod might be. If the point O were taken on the rod between the points A and B we would have, by substituting the negative limit $-y_2$,

$$R = -Km\left(\frac{1}{y_2} + \frac{1}{y_1}\right), \qquad (7)$$

and, if O were taken at the middle point of the rod, it is evident that

$$R = 0.$$

Problem 2. — Slender Rod of Uniform Section in the Form of a Circular Arc. Find the attraction of a slender rod, in the form of a circular arc, at the center of the circle; the rod being of uniform cross section (Fig. 96).

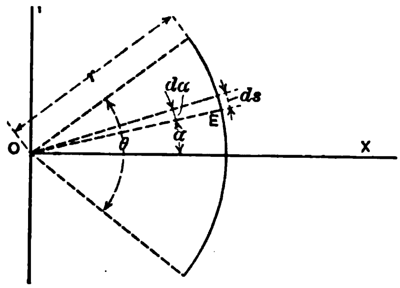

FIG. 96.

Solution. — Let r = the radius of the rod, θ = the angle which it subtends at the center and m = the mass of a unit length of the rod. Assume the axis OX, bisecting the angle θ, and let α = the angle which the radius to any point, E, makes with OX. The attraction at O of a particle at E, whose length is ds will be equal to

$$dF = \frac{Km\,ds}{r^2} = \frac{Km\,d\alpha}{r}. \qquad (1)$$

To find the resultant attraction, R, resolve the forces exerted by all the particles into components parallel to OX and OY.

Then $$\Sigma X = \frac{Km}{r}\int_{-\frac{\theta}{2}}^{\frac{\theta}{2}} \cos\alpha\,d\alpha = \frac{2\,Km}{r}\sin\frac{\theta}{2},$$

and $$\Sigma Y = \frac{Km}{r}\int_{-\frac{\theta}{2}}^{\frac{\theta}{2}} \sin\alpha\,d\alpha = 0.$$

Hence $$R = \frac{2\,Km}{r}\sin\frac{\theta}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

When $$\theta = \pi,$$

$$R = \frac{2\,Km}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If $\theta = 2\,\pi$, or the rod forms a complete circle, the attraction at the center will equal zero.

Problem 3. — Another Solution for Problem 1.

With O as a center (Fig. 95), describe an arc DGH, tangent to AD at D. Then a comparison of equation (2) (Prob. 1) and equation (1) (Prob. 2) shows that if GH is assumed to be a rod of the same mass per unit length as AB, the resultant attraction of AB at O is the same as the attraction of GH at O: since the attraction of any elementary mass, $m\,dy$, of AB is the same as that of a corresponding element, $m\,ds$, of GH, included between any two radii subtending an angle $d\alpha$ at O.

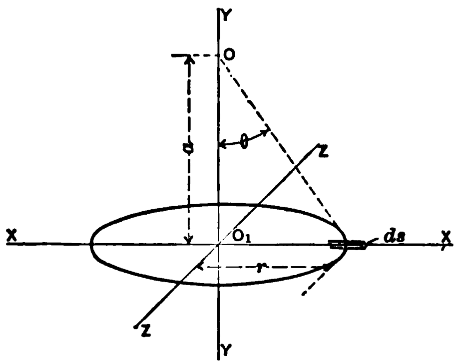

FIG. 97.

Problem 4. — Attraction of Certain Systems of Straight Rods of Equal Mass per Unit of Length.

By the analysis in Problem 3 it can easily be proved that if any three rods, of uniform mass per unit length, are placed to form a triangle, or if any number of such rods, of uniform length, are placed to form a regular polygon, their combined attraction at a point which is the center of the inscribed circle will be zero.

Problem 5. — Attraction of a Thin Circular Ring at a Point on its Axis.

Find the attraction of a thin circular ring at a point on an axis YY, perpendicular to the ring at its center, the ring being of uniform cross section (Fig. 97).

Solution. — Let a = the distance from the plane of the ring to any point O on the axis YY, r = the radius of the ring and m = the mass per unit of length. The attraction of a length ds at the point O will be

$$dF = \frac{Km\cos^2\theta}{a^2}\,ds,$$

where θ is the angle between YY and a straight line from O to a point on the ring.

Resolve dF into components parallel and perpendicular to YY, and treat all the particles of the ring in the same way.

The resultant of the components acting perpendicular to YY will equal zero (Prob. 2) and the resultant attraction at O will be equal to

$$R = \Sigma Y = \frac{Km \cos^3\theta}{a^2}\int_0^{2\pi r} ds$$

$$= \frac{Km\,2\,\pi r \cos^3\theta}{a^2} = \frac{K\,2\,\pi r\,ma}{(r^2+a^2)^{\frac{3}{2}}} = \frac{KMa}{(r^2+a^2)^{\frac{3}{2}}}, \quad \ldots \quad (1)$$

where M = the entire mass of the ring.

Problem 6. — Attraction of a Thin Circular Plate at a Point on its Axis.

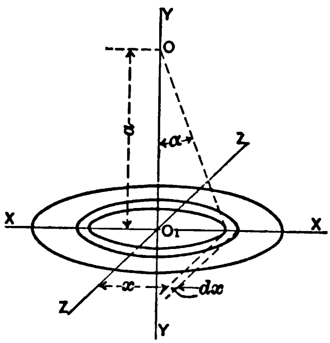

FIG. 98.

Find the attraction of a thin circular plate, of uniform thickness, at a point on the axis YY, perpendicular to the plate at its center (Fig. 98).

Solution. — Let a = the distance from the plate to the point O on an axis YY and r = the radius of the plate.

Let d = the mass per unit of volume; that is, the density of the plate, and let t = its thickness, which is very small. Divide the plate into a series of elementary rings of radius x and width dx and let α = the angle between YY and a straight line from O to a point on one of the elementary rings.

From equation (1) (Prob. 5) we find the attraction at O of one of these elementary rings to be

$$dF = \frac{Kd\,2\,\pi t \cos^3\alpha\, x\, dx}{a^2}. \quad \ldots \quad (1)$$

But $\qquad x = a \tan\alpha,$

and hence $\qquad dx = a\dfrac{d\alpha}{\cos^2\alpha}.$

Substituting these values in equation (1) and integrating we find the resultant attraction at O of the entire plate to be

$$R = Kd\,2\,\pi t\int_0^{\theta} \sin\alpha\, d\alpha = Kd\,2\,\pi t\,(1 - \cos\theta). \quad \ldots \quad (2)$$

If $\qquad M$ = the entire mass of the plate,

$$R = \frac{2\,KM\,(1-\cos\theta)}{r^2}. \quad \ldots \quad (3)$$

Problem 7. — Attraction of a Right Circular Cone at a Point at its Apex.

Find the attraction of a right circular cone at a point at its apex.

Solution. — It is evident from equation (2) (Prob. 6) that the attraction at O of all homogeneous thin circular plates of equal density and thickness, which are similarly situated with respect to O, will be the same, provided the angles θ subtended at O are the same.

Hence it follows that the attraction at a point on the apex of a right circular cone, if we let d = the density, will be equal to

$$R = Kd\,2\,\pi\,(1 - \cos\theta)\int_0^h dy$$
$$= Kd\,2\,\pi\,h\,(1 - \cos\theta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where h = the altitude of the cone and θ = the half angle subtended at the vertex.

Problem 8. — Attraction of a Right Circular Cylinder at a Point on its Axis.

Find the attraction of a right circular cylinder at a point on its axis.

Solution. — Let r = the radius of the cylinder and x_1 and x_2 the distances of the point O from the ends of the cylinder and let $x_1 > x_2$. Let d = the density. Then from equation (2) (Prob. 6), we get for the attraction of a thin slice of thickness dx, perpendicular to the axis of the cylinder at a distance x from O,

$$dF = Kd\,2\,\pi\,(1 - \cos\theta)\,dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where
$$\cos\theta = \frac{x}{\sqrt{r^2 + x^2}}.$$

Hence
$$R = Kd\,2\,\pi\int_{x_2}^{x_1}\left(1 - \frac{x}{\sqrt{r^2 + x^2}}\right)dx$$

$$= 2\,Kd\pi\left[x - \sqrt{r^2 + x^2}\right]_{x_2}^{x_1} = 2\,Kd\pi\left[x_1 - x_2 - \sqrt{r^2 - x_1^2} + \sqrt{r^2 + x_2^2}\right]. \quad (2)$$

If the point O is at the center of one end of the cylinder, $x_2 = 0$, and equation (2) becomes

$$R = 2\,Kd\pi\,(x_1 + r - \sqrt{r^2 + x_1^2}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the point O is taken on the axis between the ends of the cylinder, the attraction may be found by substituting a negative limit $(-\,x_2)$, when integrating equation (2), which would give

$$R = 2\,Kd\pi\left[x_1 + x_2 - \sqrt{r^2 + x_1^2} + \sqrt{r^2 + x_2^2}\right]. \quad . \quad . \quad . \quad . \quad (4)$$

If O were taken at the middle point of the axis of the cylinder, it is evident that equation (4) would give $R = 0$.

Problem 9. — Attraction of a Hollow Circular Cylinder at a Point on Its Axis.

Find the attraction of a hollow right circular cylinder at a point on its axis.

Solution. — In this case the attraction may be determined by finding the difference of the attractions of the two solid cylinders, of radii equal to the outside and inside radii of the hollow cylinder, by means of the formulas (Prob. 8).

If the thickness is very small another method of determining the attraction would be the following:

Divide the cylinder into thin rings by taking sections perpendicular to its axis. Let r = the radius and t = the thickness of the cylinder, dx = the width of one of the elementary rings and x = its distance from the point O, on the axis of the cylinder.

Then from equation (1) (Prob. 5) the attraction of the elementary ring at O will be equal to

$$dF = \frac{Kd\,2\,\pi r t x\,dx}{(r^2 + x^2)^{\frac{3}{2}}},$$

and, if we let x_1 and x_2 be the distances of the ends of the cylinder from O, the attraction of the entire cylinder will be

$$R = Kd\,2\,\pi rt\int_{x_2}^{x_1}\frac{x\,dx}{(r^2+x^2)^{\frac{3}{2}}}$$

$$= Kd\,2\,\pi rt\left[\frac{1}{\sqrt{r^2+x_2^2}}-\frac{1}{\sqrt{r^2+x_1^2}}\right].$$

Problem 10. — Attraction of a Sphere at a Point.

Find the attraction of a sphere at any point O: (a) outside of the sphere, (b) on its surface, (c) inside of the sphere (Fig. 99).

Solution. — (a) *Point Outside of Sphere.* — Assume the axis OX through the point O and the center of the sphere and let a = the distance OA. Let r = the radius of the sphere and d = its density. Consider the sphere to be made up of thin slices of thickness dx, perpendicular to OX. Let y = the radius of one of these slices, whose distance from O is equal to x. From equation (2) (Prob. 6), we find the attraction of this slice at O to be

$$dF = Kd\,2\,\pi\,(1-\cos\theta)\,dx$$

$$= Kd\,2\,\pi\left(1-\frac{x}{\sqrt{x^2+y^2}}\right)dx.$$

But $\quad y^2 = r^2-(x-a)^2 \quad\text{and}\quad x^2+y^2 = r^2-a^2+2\,ax;$

hence
$$dF = Kd\,2\,\pi\left[1-\frac{x}{(r^2-a^2+2\,ax)^{\frac{1}{2}}}\right]dx. \qquad (1)$$

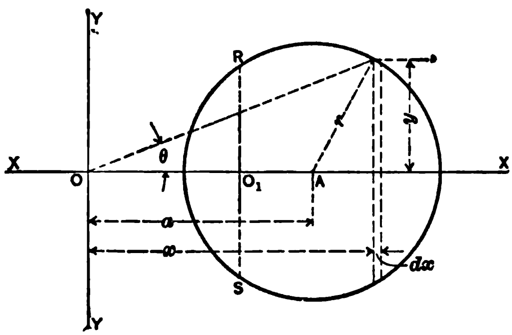

FIG. 99.

The resultant attraction of the entire sphere at O will be equal to

$$R = Kd\,2\,\pi\int_{a-r}^{a+r}\left[1-\frac{x}{(r^2-a^2+2\,ax)^{\frac{1}{2}}}\right]dx. \qquad (2)$$

Integrating,

$$R = Kd\,2\,\pi\left[x-\frac{x}{a}(r^2-a^2+2\,ax)^{\frac{1}{2}}+\frac{1}{3\,a^2}(r^2-a^2+2\,ax)^{\frac{3}{2}}\right]_{a-r}^{a+r} \qquad (3)$$

$$= Kd\,2\,\pi\left[2\,r-\frac{(a+r)^2}{a}+\frac{(a-r)^2}{a}+\frac{(a+r)^3}{3\,a^2}-\frac{(a-r)^3}{3\,a^2}\right],$$

which readily reduces to

$$R = \frac{4\,Kd\,\pi r^3}{3\,a^2}. \qquad (4)$$

If M = the mass of the entire sphere,

$$R = \frac{KM}{a^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which shows that the attraction of the sphere on a particle, at a point outside the sphere, is the same as if the entire mass of the sphere were concentrated at its center.

(b) *Point on the Surface of the Sphere.* — In this case $a = r$ and equation (1) becomes

$$R = Kd\,2\pi \int_0^{2r} \left[1 - \left(\frac{x}{2r}\right)^{\frac{1}{2}}\right] dx \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Integrating,

$$R = Kd\,2\pi \left[x - \frac{2}{3}\,\frac{x^{\frac{3}{2}}}{(2r)^{\frac{1}{2}}}\right]_0^{2r},$$

which reduces to

$$R = \frac{4\,Kd\,\pi r}{3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which could have been obtained by substituting $a = r$ in equation (4). It is evident that the attraction in this case varies directly as the radius of the sphere.

(c) *Point Inside of the Surface of the Sphere.* — Let O_1 (Fig. 99) be any point within the sphere and RS a plane perpendicular to OX through O_1, and let $a = O_1A$, the distance of O_1 from the center of the sphere.

The attraction at O_1 of an elementary slice of the larger segment, perpendicular to OX, whose distance from O_1 equals x and radius equals y, will evidently be given by equation (1), and the resultant attraction of the segment will be

$$R_1 = Kd\,2\pi \int_0^{r+a} \left[1 - \frac{x}{(r^2 - a^2 + 2\,ax)^{\frac{1}{2}}}\right] dx.$$

Hence, substituting the limits in equation (3), we have

$$R_1 = Kd\,2\pi \left[r + a - \frac{(r+a)^2}{a} + \frac{(r+a)^3}{3\,a^2} - \frac{(r^2 - a^2)^{\frac{3}{2}}}{3\,a^2}\right],$$

which reduces to

$$R_1 = \frac{Kd\,2\pi}{3\,a^2}\left[r^3 + a^3 - (r^2 - a^2)^{\frac{3}{2}}\right]. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Calling x positive to the left, the attraction at O_1 of an elementary slice of the smaller segment, whose distance from O_1 equals x and radius equals y, will be equal to

$$dF = K\,2\pi d\left[1 - \frac{x}{\sqrt{x^2 + y^2}}\right] dx = Kd\,2\pi\left[1 - \frac{x}{(r^2 - a^2 - 2\,ax)^{\frac{1}{2}}}\right] dx.$$

Integrating, the resultant attraction of the smaller segment will be

$$R_2 = Kd\,2\pi \int_0^{r-a} \left[1 - \frac{x}{(r^2 - a^2 - 2\,ax)}\right] dx$$

$$= Kd\,2\pi \left[x + \frac{x}{a}(r^2 - a^2 - 2\,ax)^{\frac{1}{2}} - \frac{1}{3\,a^2}(r^2 - a^2 - 2\,ax)^{\frac{3}{2}}\right]_0^{r-a}$$

$$= \frac{Kd\,2\pi}{3\,a^2}\left[a^3 - r^3 + (r^2 - a^2)^{\frac{3}{2}}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The resultant attraction of the two segments at O_1 will evidently be equal to the algebraic sum of R_1 and R_2, and will be equal to

$$R = \frac{4\,Kd\pi a}{3}. \qquad (10)$$

Therefore, the attraction varies as the distance of the point from the center of the sphere and, at any point within the sphere, is the same as would be exerted by a sphere of equal density whose radius is equal to a; becoming zero at the center of the sphere. The shell of thickness $(r - a)$ exerts no attraction at the point and therefore the attraction of any hollow sphere at a point on its inside surface equals zero.

Problem 11. — Attraction of a Hollow Sphere at a Point.

Find the attraction of a thin hollow sphere at any point O: (a) outside of the sphere, (b) on its surface, (c) inside of the sphere.

Solution. — The attraction of a hollow sphere at any point will evidently be equal to the difference of the attractions of two concentric, solid spheres of the same density, whose radii are equal to the outside and inside radii of the hollow sphere.

(a) *Point Outside of the Sphere.* — Let r_1 = the outside radius and r_2 = the inside radius of the sphere. From equation (4) (Prob. 10), using the same notation as before, we find that the resultant attraction

$$R = \frac{4\,Kd\pi\,(r_1^3 - r_2^3)}{3\,a^2}. \qquad (1)$$

If we let t = the thickness of the sphere and $r = r_1$, we shall have $r_2 = r - t$,

and $$(r_1^3 - r_2^3) = (3\,r^2t - 3\,rt^2 + t^3) = t\,(3\,r^2 - 3\,rt + t^2). \qquad (2)$$

For a sphere of very small thickness we may write equation (2), without an appreciable error,

$$(r_1^3 - r_2^3) = 3\,r^2t.$$

Substituting this value in (1),

$$R = \frac{Kd\,4\,\pi r^2t}{a^2} = \frac{KM}{a^2}, \qquad (3)$$

where M equals the entire mass of the sphere. Hence, as in the case of the solid sphere, the attraction is the same as though the mass of the sphere were concentrated at its center.

(b) *Point on the Surface.* — Substituting $a = r$ in equation (3) we have

$$R = Kd\,4\,\pi t = \frac{KM}{r^2}. \qquad (4)$$

(c) *Point within the Sphere.* — Since the attraction of a solid sphere, at a point inside the sphere, depends only on the distance of the point from the center and is independent of the radius of the sphere (Prob. 10), the attraction of any hollow sphere, at a point within its inside surface, will be equal to zero; for it will be equal to the difference of the attractions of two solid spheres, of radii equal respectively to the outside and inside radii of the hollow sphere, and these attractions will be the same for any point, whose distance from their center is less than the inside radius of the hollow sphere.

A simple geometrical proof in the case of a thin hollow sphere is the following:

Let O be any point inside of the sphere and let BE be any line through O, cutting the surface at the points B and E, which are at the respective distances b and e from O (Fig. 100). With BE as an axis, describe a cone with a very

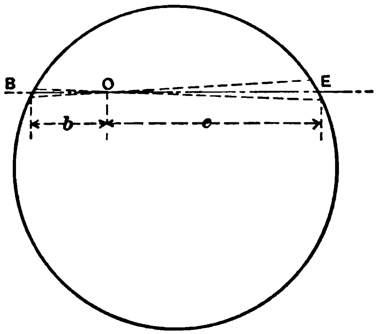

FIG. 100.

small angle at the vertex. The parts of the cone, BO and OE, between the surface of the sphere and the vertex, are evidently similar and hence the areas, cut from the surface at B and E, will be proportional to the squares of their distances from O. Let m_1 = mass of the element, cut from the shell at B, and m_2, that at E.

Then
$$\frac{m_1}{b^2} = \frac{m_2}{e^2}.$$

Hence
$$K\frac{m_1}{b^2} = K\frac{m_2}{e^2},$$

and the attractions of the masses m_1 and m_2 are equal and opposite. Since the whole surface can be divided in the same way, by a series of elementary cones with vertices at O, the resultant attraction of the entire sphere at O will be equal to zero.

Note.—If a system of lines is constructed in a field of force, in such a manner that the tangent at any point in any one of them is in the direction of the attraction at that point, the lines are called *lines of force*. Thus in Prob. 1 (Fig. 95), the attraction at any point O bisects the angle AOB and hence is tangent to the hyperbola passing through O and having A and B for its foci. Therefore, the lines of force, for the attraction of the rod AB, will be a system of hyperbolas with A and B as foci. In the case of the sphere, the lines of force will evidently be straight lines radiating from the center of the sphere.

87. Determination of the Constant K.—In Problem 10 (Art. 86), we proved that the attraction of a sphere is the same as if its mass were concentrated at its center, the attraction of a sphere on a unit mass at a distance a from its center being equal

to $\frac{KM}{a^2}$, for both the solid and the hollow sphere. Hence the formula for the attraction of two particles,

$$F = \frac{Kmm_1}{c^2} \text{ (Art. 85),}$$

will apply in the case of two spheres, which are homogeneous throughout, or made up of a series of homogeneous shells; m and m_1 being the masses of the spheres and c the distance between their centers.

By physical experiment the force of attraction between two spheres has been measured directly and the value of K, in the C.G.S. system of units, found to be equal to 0.00000006658, which is the force in dynes with which two homogeneous spheres, whose masses are 1 gram each, attract each other when the distance between their centers is 1 centimeter.

It is evident that the formula for acceleration,

$$\frac{F}{m_1} = \frac{Km}{c^2} \text{ (Art. 85),}$$

will also apply in the case of two spheres of masses m and m_1. The attraction exerted by the Earth on a unit mass at its surface will be $R = \frac{4}{3} Kd\pi r$ (Prob. 10, Art. 86), very nearly, and hence the acceleration produced by this attraction on a freely falling weight will be

$$g = \frac{4}{3} Kd\pi r.$$

When g, K and r, the radius of the Earth, are known, this formula will give the value of the mean density of the Earth

$$d = \frac{3\,g}{4\,K\pi r}.$$

Substituting values for latitude 45°, $g = 980.63$ cm., $r =$ 636,700,000 cm., and for K, the value given above, we find

$$d = 5.52.$$

CHAPTER III.

CENTER OF GRAVITY.*

88. Center of a System of Parallel Forces. — The center of a system of parallel forces is the point through which the line of action of the resultant always passes, no matter how the forces are turned, provided only: —

(1) Their points of application remain unchanged.

(2) Their relative magnitudes remain unchanged.

(3) They remain parallel to each other.

Let F, F_1 and F_2 represent any system of parallel forces (Fig. 101). Refer the system to rectangular coördinate axes, OX, OY

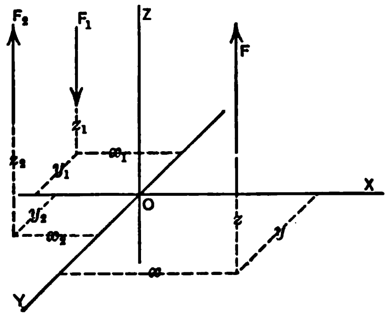

FIG. 101.

and OZ, choosing the axes so that the forces are parallel to the axis OZ. Let the coördinates of the points of application of the forces be (x, y, z), (x_1, y_1, z_1) and (x_2, y_2, z_2).

Then, if R equals the resultant of the system of forces and (x_0, y_0) the coördinates of its point of application with respect to the X and Y planes, we have (Art. 69)

$$R = \Sigma F, \qquad x_0 = \frac{\Sigma Fx}{\Sigma F}, \qquad y_0 = \frac{\Sigma Fy}{\Sigma F}.$$

Imagine the forces turned parallel to the axis OX, the points of application remaining unchanged. Then, if (y_0, z_0) are the

* Frequently designated by the term *centroid*, when applied to lines, areas and volumes.

coördinates of the point of application of R with respect to the Y and Z planes, we have

$$R = \Sigma F, \qquad y_0 = \frac{\Sigma Fy}{\Sigma F}, \qquad z_0 = \frac{\Sigma Fz}{\Sigma F},$$

and if the forces are turned parallel to OY, the points of application remaining unchanged, we have

$$R = \Sigma F, \qquad x_0 = \frac{\Sigma Fx}{\Sigma F}, \qquad z_0 = \frac{\Sigma Fz}{\Sigma F}.$$

Therefore there is one point, called the *center of the system*, whose coördinates are

$$x_0 = \frac{\Sigma Fx}{\Sigma F}, \qquad y_0 = \frac{\Sigma Fy}{\Sigma F}, \qquad z_0 = \frac{\Sigma Fz}{\Sigma F},$$

and through which the resultant always passes, provided the three conditions stated above are satisfied.

When $\Sigma F = 0$; x_0, y_0 and z_0 become infinite and therefore such a system of forces has no center. In other words, if $\Sigma F = 0$, the resultant is a couple.

89. Center of Gravity of a Body. — Any body may be conceived to be divided into parts, or particles, of greater or less extent, on each one of which gravity exerts a force equal to its weight; the forces exerted on the separate parts being practically parallel to each other. The *center of gravity* of the body is the center of this system of parallel forces. Hence, if w is the weight per unit volume of a body, and dV is the volume of any elementary particle, the force exerted by gravity on the particle will be $w\,dV$. Substituting in the formulas (Art. 88), we have

$$W = \int w\,dV, \quad x_0 = \frac{\int wx\,dV}{\int w\,dV}, \quad y_0 = \frac{\int wy\,dV}{\int w\,dV}, \quad z_0 = \frac{\int wz\,dV}{\int w\,dV},$$

where W denotes the entire weight of the body, and (x_0, y_0, z_0), the coördinates of its center of gravity.

By transposing we obtain the quantities

$$\int wx\,dV = x_0W, \int wy\,dV = y_0W, \int wz\,dV = z_0W$$

which are called the *moments of the weight* of the body with respect to the X, Y and Z planes, respectively.

In computing the moment of a weight with respect to a plane we will call the moment positive, if the weight is situated on the positive side of the plane, and negative if on the other side.

90. Center of Gravity of a Homogeneous Body. — If a body is homogeneous, w = a constant and the equations (Art. 89) reduce to

$$W = w\int dV, \quad x_0 = \frac{\int x\,dV}{\int dV}, \quad y_0 = \frac{\int y\,dV}{\int dV}, \quad z_0 = \frac{\int z\,dV}{\int dV}.$$

If we let V = the total volume of the body, the quantities

$$\int x\,dV = x_0 V, \quad \int y\,dV = y_0 V, \quad \int z\,dV = z_0 V$$

will equal the *moments of the volume* with respect to the X, Y and Z planes, respectively.

91. Center of Gravity of a System of Bodies. — If, instead of a single body, we have a system of bodies whose weights are W_1, W_2, W_3, etc., the coördinates of their centers of gravity being respectively (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), etc.: and, if we denote by (x_0, y_0, z_0) the coördinates of the center of gravity of the system and by W its total weight, we have

$$W = W_1 + W_2 + W_3 + \text{etc.}$$

The algebraic sum of the moments of the weights with respect to the X plane will be

$$W_1x_1 + W_2x_2 + W_3x_3 + \text{etc.} = \Sigma Wx \text{ (Art. 89).}$$
$$\text{But } \Sigma Wx = Wx_0 \text{ (Art. 88).}$$

Similarly, the algebraic sum of the moments of the weights with respect to the Y and Z planes will be

$$\Sigma Wy = Wy_0 \quad \text{and} \quad \Sigma Wz = Wz_0.$$

Hence $$x_0 = \frac{\Sigma Wx}{W}, \quad y_0 = \frac{\Sigma Wy}{W}, \quad z_0 = \frac{\Sigma Wz}{W}.$$

92. Center of Mass. — Since the masses of the particles of a body are directly proportional to their weights, the *center of mass* will be the same as the center of gravity.

If we let M denote the entire mass of a body and dM the mass of a particle whose volume is dV, then,

$$dM = \frac{W\,dV}{g}.$$

Substituting this value in the equations (Art. 89) we have

$$M = \int dM, \quad x_0 = \frac{\int x\,dM}{\int dM}, \quad y_0 = \frac{\int y\,dM}{\int dM}, \quad z_0 = \frac{\int z\,dM}{\int dM}.$$

93. Center of Gravity of a Slender Rod of Uniform Section and Material. — If the rod is straight it is evident that its center of gravity will be at the middle point of its center line. If the center line of the rod is a plane curve the center of gravity may be found as follows:

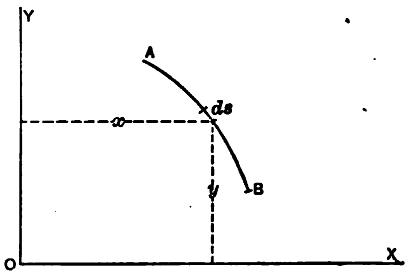

FIG. 102.

Let AB (Fig. 102) be the center line of a slender rod of uniform section and material, whose weight per unit of length is equal to w. Refer the rod to any pair of rectangular coördinate axes, OX and OY, in the plane of AB. The weight of an elementary length ds will be $w\,ds$ and by substituting this value for $w\,dV$ in the equations for the coördinates of the center of gravity (Art. 89), we have

$$x_0 = \frac{\int wx\,ds}{\int w\,ds} = \frac{\int x\,ds}{\int ds},$$

$$y_0 = \frac{\int wy\,ds}{\int w\,ds} = \frac{\int y\,ds}{\int ds},$$

$$z_0 = 0.$$

If we let L equal the length of the center line of the rod, it is evident that

$$x_0L = \int x\,ds \quad \text{and} \quad y_0L = \int y\,ds.$$

These quantities are called the *moments of the line*, with respect to the axes OY and OX, respectively. If the center line of the rod is not a plane curve it is necessary to find three coördinates of the center of gravity. By substituting in the formulas (Art. 89) as before, these will evidently be equal to

$$x_0 = \frac{\int x\,ds}{\int ds}, \quad y_0 = \frac{\int y\,ds}{\int ds}, \quad z_0 = \frac{\int z\,ds}{\int ds},$$

The moments of the line with respect to the X, Y and Z planes will be respectively

$$x_0 L = \int x\,ds, \quad y_0 L = \int y\,ds, \quad z_0 L = \int z\,ds.$$

94. Center of Gravity of a Line. — By the expression, *center of gravity of a line*, we mean the point which is the center of gravity of a slender rod of uniform section and material, of which the line is the center line. The methods for determining the coördinates of the center of gravity are, therefore, those given in Art. 93.

Hence, to find the coördinates of the center of gravity of a line consider it to be divided into small elementary lengths and divide the algebraic sums of the moments of these lengths about the coördinate axes by the length of the line.

The moment of a line with respect to an axis or plane will be positive if the line is on the positive side of the axis or plane; and negative, if otherwise.

It is evident that the moment of a line about an axis passing through its center of gravity will be equal to zero.

95. Center of Gravity of a System of Lines. — Any system of lines may be treated in the same manner as a system of weights and the coördinates of the center of gravity found by the method stated in Art. 91. If the lines are all in the same plane and we let L_1, L_2, L_3, etc., equal their lengths and (x_1, y_1), (x_2, y_2), (x_3, y_3), etc., the coördinates of their respective centers of gravity, we have, for the algebraic sum of their moments about OY,

$$L_1x_1 + L_2x_2 + L_3x_3 + \text{etc.} = \Sigma Lx,$$

and about OX,

$$L_1y_1 + L_2y_2 + L_3y_3 + \text{etc.} = \Sigma Ly.$$

If we let the sum of the lengths equal L,

$$x_0 = \frac{\Sigma Lx}{L} \quad \text{and} \quad y_0 = \frac{\Sigma Ly}{L}.$$

If the lines are not all in the same plane, the three coördinates of the center of gravity of the system,

$$x_0 = \frac{\Sigma Lx}{L}, \qquad y_0 = \frac{\Sigma Ly}{L}, \qquad z_0 = \frac{\Sigma Lz}{L},$$

can be found in a similar manner.

Hence the coördinates of the center of gravity of any line, which can be divided into parts whose centers of gravity are known, may be found by dividing the algebraic sums of the moments of the parts, with respect to any convenient set of coördinate axes, by the total length of the line.

96. Problems. — Centers of Gravity of Lines. — The following problems can be solved by the methods given in Arts. 94 and 95.

The lines may be taken to represent slender rods of uniform section and material, or very narrow areas of uniform width.

Problem 1. — Circular Arc.

Let AB be a circular arc, whose radius $= r$, and let OX and OY be two coördinate axes with an origin at O, the center of curvature (Fig. 103). Let θ_1 and θ_2 be the angles between OX and the radii OA and OB to the ends of the arc. Find the coördinates of the center of gravity with respect to OX and OY.

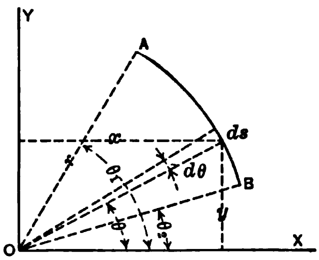

Fig. 103.

Solution. — The coördinates of the center of gravity can be found from the equations,

$$x_0 = \frac{\int x\,ds}{\int ds} \quad \text{and} \quad y_0 = \frac{\int y\,ds}{\int ds} \text{ (Art. 93).}$$

Using polar coördinates, $x = r\cos\theta$, $y = r\sin\theta$ and $ds = r d\theta$.

Hence
$$x_0 = \frac{\int_{\theta_2}^{\theta_1} r^2 \cos\theta\, d\theta}{\int_{\theta_2}^{\theta_1} r\, d\theta} = \frac{r\sin\theta \Big|_{\theta_2}^{\theta_1}}{\theta \Big|_{\theta_2}^{\theta_1}} = \frac{r(\sin\theta_1 - \sin\theta_2)}{\theta_1 - \theta_2},$$

and
$$y_0 = \frac{\int_{\theta_2}^{\theta_1} r^2 \sin\theta\, d\theta}{\int_{\theta_2}^{\theta_1} r\, d\theta} = \frac{-r\cos\theta \Big|_{\theta_2}^{\theta_1}}{\theta \Big|_{\theta_2}^{\theta_1}} = \frac{r(\cos\theta_2 - \cos\theta_1)}{\theta_1 - \theta_2}.$$

When $\theta_2 = 0$, the equations reduce to

$$x_0 = \frac{r \sin \theta_1}{\theta_1}, \quad y_0 = \frac{r(1 - \cos \theta_1)}{\theta_1}.$$

Problem 2.

Find the coördinates x_0 and y_0 (Prob. 1),

(a) when $\theta_1 = \theta_1$, $\theta_2 = -\theta_1$.

(b) when $\theta_1 = 90°$, $\theta_2 = 0$.

Problem 3. — Semicircular Arc.

Show that the center of gravity of the semicircular arc of radius r is on a radius bisecting the arc, and that its distance from the center is equal to $\frac{2r}{\pi}$.

Problem 4. — Sections Represented by Lines.

Find the coördinates of the center of gravity of the sections shown in Figs. 104a, b, c, d.

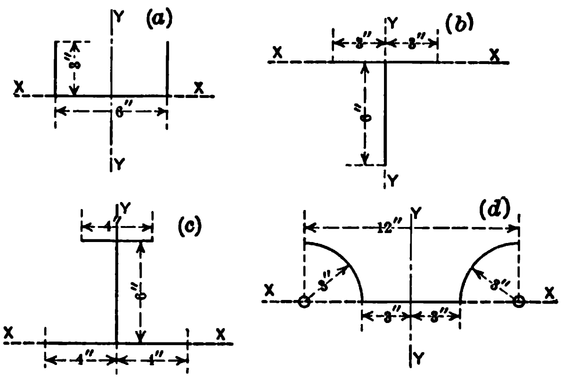

FIG. 104.

Problem 5.

Find the coördinates of the center of gravity of the section lying on the right of the axis YY (Fig. 104d).

97. Center of Gravity of a Thin Flat Plate of Uniform Thickness and Material. — Let abc (Fig. 105) represent a homogeneous thin flat plate of uniform thickness, t. Refer the plate to the coördinate axes OX and OY, in the plane of its middle layer. If we let dA equal the area of the base of a prismatic particle, whose coördinates are (x, y), the volume dV, of the elementary particle, will be equal to $t\,dA$ and the formulas for the coördinates of the center of gravity of a homogeneous body (Art. 90) will reduce to

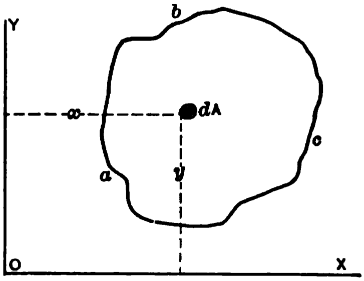

FIG. 105.

$$x_0 = \frac{\int xt\,dA}{\int t\,dA} = \frac{\int x\,dA}{\int dA},$$

and $$y_0 = \frac{\int yt\,dA}{\int t\,dA} = \frac{\int y\,dA}{\int dA},$$

the coördinate z_0 being zero.

If we let A equal the area of the middle layer it is evident that

$$x_0 A = \int x\,dA \quad \text{and} \quad y_0 A = \int y\,dA.$$

These quantities are called the *moments of the area, A,* with respect to the axes OY and OX.

98. Center of Gravity of a Plane Surface. — By the center of gravity of a plane surface we mean that point which is the center of gravity of a thin flat plate of uniform thickness and material, whose middle layer is the surface in question. The formulas for its coördinates are, therefore, those in Art. 97.

$$x_0 = \frac{\int x\,dA}{A} \quad \text{and} \quad y_0 = \frac{\int y\,dA}{A}.$$

Hence, to find the coördinates of the center of gravity of a plane surface, consider it to be divided into small elementary areas and divide the algebraic sums of the moments of the areas, about the coördinate axes, by the total area of the surface.

As in the case of the line, the *moment of an area with respect to an axis will be positive if the area is on the positive side of the axis and negative if otherwise. It is evident that the moment of an area about an axis passing through its center of gravity will be equal to zero.*

99. Center of Gravity of a Group of Plane Surfaces. — Like the system of lines (Art. 95) any group of plane surfaces may be treated in the same manner as a system of weights and the coördinates of the center of gravity found by the method in Art. 91.

Let A_1, A_2, A_3, etc., be the areas of any group of surfaces in the

same plane and let the coördinates of their centers of gravity be (x_1, y_1), (x_2, y_2), (x_3, y_3), etc.

The algebraic sum of their moments with respect to OY will be

$$A_1x_1 + A_2x_2 + A_3x_3 + \text{etc.} = \Sigma Ax,$$

and with respect to OX

$$A_1y_1 + A_2y_2 + A_3y_3 + \text{etc.} = \Sigma Ay.$$

If we let the sum of the areas equal A,

$$x_0 = \frac{\Sigma Ax}{A} \quad \text{and} \quad y_0 = \frac{\Sigma Ay}{A}.$$

Hence the coördinates of the center of gravity of any plane surface which can be divided into parts, whose centers of gravity are known, can be found by dividing the algebraic sum of the moments of the parts about any convenient set of coördinate axes by the total area of the surface.

100. Centers of Gravity of Surfaces not in the Same Plane. — By a method similar to that in Arts. 97–99 we can show that the coördinates of the center of gravity of any surface not necessarily plane, with respect to any three rectangular axes, will be

$$x_0 = \frac{\int x\,dA}{A}, \quad y_0 = \frac{\int y\,dA}{A} \quad \text{and} \quad z_0 = \frac{\int z\,dA}{A},$$

and for any group of surfaces, the coördinates of whose centers of gravity are known,

$$x_0 = \frac{\Sigma Ax}{A}, \quad y_0 = \frac{\Sigma Ay}{A} \quad \text{and} \quad z_0 = \frac{\Sigma Az}{A}.$$

101. Problems. — Centers of Gravity of Plane Surfaces. — The following problems are given to illustrate the methods of finding the centers of gravity of plane surfaces. In cases where the methods of calculus are used, the areas may be considered to be divided into elementary strips, parallel to one of the coördinate axes; the moment of any strip about either axis being equal to the product of its area and the distance of its center from the axis.

Problem 1. — Rectangle and Parallelogram.

Prove that the perpendicular distance of the center of gravity of a rectangle, or parallelogram, from its base is equal to one half its altitude.

Solution. — Let b and h equal the base and altitude of the rectangle, or parallelogram (Fig. 106). Let the axis OX coincide with the base. Take

for the elementary area a strip parallel to OX whose width is dy. Then $dA = b\,dy$ and (Art. 98)

$$y_0 = \frac{\int y\,dA}{A} = \frac{b\int_0^h y\,dy}{bh} = \frac{h}{2},$$

which is true for either the rectangle or the parallelogram.

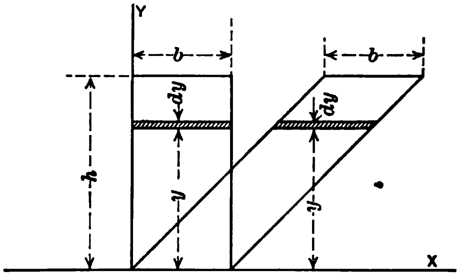

FIG. 106.

Problem 2. — Triangle.

Find the perpendicular distance of the center of gravity of a triangle from its base.

Let b and h equal the base and altitude, respectively, and assume an axis OX through the vertex, parallel to the base (Fig. 107)

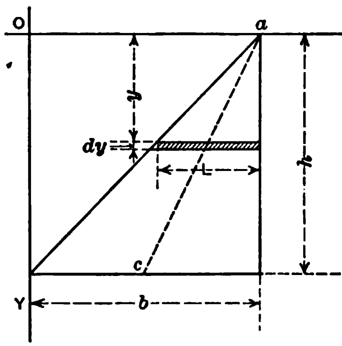

FIG. 107.

Solution. — Let L = the length of an elementary strip, parallel to and at a distance y from OX. Then $dA = L\,dy$. But $L : y = b : h$; hence

$$dA = \frac{b}{h} y\,dy.$$

Therefore,
$$y_0 = \frac{\int y\,dA}{A} = \frac{\frac{b}{h}\int_0^h y^2\,dy}{\frac{bh}{2}} = \frac{2}{3}h.$$

Hence the perpendicular distance of the center of gravity from the base will be equal to $\frac{h}{3}$ and, since the center of gravity of each elementary strip will be its middle point, the center of gravity of the triangle will be on the median line, *ac*.

Problem 3. — Parabolic Half Segment.

Find the coördinates of the center of gravity of the parabolic half segment, bounded by the axis of the parabola and the ordinate to the point on the curve whose coördinates are (x_1, y_1) (Fig. 108).

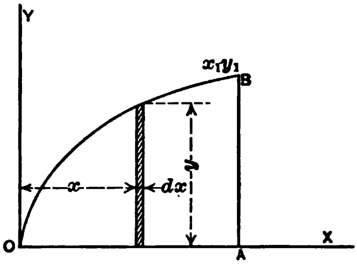

FIG. 108.

Solution. — The equation of the parabola referred to its axis OX and the axis OY, through the vertex, will be $y^2 = cx$. Take for dA an elementary strip, parallel to OY, whose width is dx and distance from OY is equal to x.

Then $\qquad dA = y\,dx \quad$ and $\quad x_0 = \dfrac{\int x\,dA}{\int dA} = \dfrac{\int_0^{x_1} xy\,dx}{\int_0^{x_1} y\,dx}$

$$= \frac{c^{\frac{1}{2}}\int_0^{x_1} x^{\frac{3}{2}}\,dx}{c^{\frac{1}{2}}\int_0^{x_1} x^{\frac{1}{2}}\,dx} = \frac{3}{5}x_1.$$

In a similar manner, $\quad y_0 = \dfrac{\int y\,dA}{\int dA} = \dfrac{\int_0^{x_1}\frac{y}{2}y\,dx}{\int_0^{x_1} y\,dx}$

$$= \frac{\frac{c}{2}\int_0^{x_1} x\,dx}{c^{\frac{1}{2}}\int_0^{x_1} x\ \ dx} = \frac{3}{8}c^{\frac{1}{2}}\,x_1^{\frac{1}{2}} = \frac{3}{8}y_1.$$

The problem might have been solved by taking strips parallel to OX, in which case $dA = (x_1 - x)\,dy$ and

$$y_0 = \frac{\int y\,dA}{\int dA} = \frac{\int_0^{y_1} y\,(x_1 - x)\,dy}{\int_0^{y_1}(x_1 - x)\,dy} = \frac{\int_0^{y_1}\left(\frac{yy_1^2}{c} - \frac{y^3}{c}\right)dy}{\int_0^{y_1}\left(\frac{y_1^2}{c} - \frac{y^2}{c}\right)dy} = \frac{3}{8}y_1.$$

The coördinate x_0 might also be found by dividing the area into strips parallel to OX.

Problem 4. — Circular Sector.

Find the coördinates of the center of gravity of the circular sector, subtending the angle θ_1, with respect to the axes OX and OY (Fig. 109).

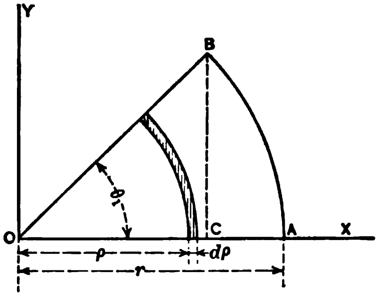

FIG. 109.

Solution. — In this problem, instead of dividing the area into strips parallel to OX or OY, it will be found more convenient to consider it as made up of circular strips, with centers at O. If we let ρ = the radius, and $d\rho$ = the width of one of these strips, its area $dA = \theta_1\rho\, d\rho$ and the distances of its center of gravity from OY and OX will be

$$\frac{\rho \sin\theta_1}{\theta_1} \text{ and } \frac{\rho(1-\cos\theta_1)}{\theta_1}, \text{ respectively (Prob. 1).}$$

Then
$$x_0 = \frac{\int x\,dA}{\int dA} = \frac{\sin\theta_1 \int_0^r \rho^2\,d\rho}{\theta_1 \int_0^r \rho\,d\rho} = \frac{2}{3}r\frac{\sin\theta_1}{\theta_1},$$

and
$$y_0 = \frac{\int y\,dA}{\int dA} = \frac{\rho(1-\cos\theta_1)\int_0^r \rho^2\,d\rho}{\theta_1 \int_0^r \rho\,d\rho} = \frac{2}{3}r\frac{1-\cos\theta_1}{\theta_1}.$$

Another method of solving this problem would be to consider the area to be made up of elementary sectors, subtending the angle $d\theta$ at O. Let θ = the angle between OX and one of these sectors. Its area will be $dA = \frac{r^2\,d\theta}{2}$ and the coördinates of its center of gravity will be $x = \frac{2}{3}r\cos\theta$ and $y = \frac{2}{3}r\sin\theta$ (Prob. 2).

Then
$$x_0 = \frac{\frac{r^3}{3}\int_0^{\theta_1}\cos\theta\,d\theta}{\frac{r^2}{2}\int_0^{\theta_1}d\theta} = \frac{2}{3}r\frac{\sin\theta_1}{\theta_1},$$

and
$$y_0 = \frac{\frac{r^3}{3}\int_0^{\theta_1}\sin\theta\,d\theta}{\frac{r^2}{2}\int_0^{\theta_1}d\theta} = \frac{2}{3}r\frac{1-\cos\theta_1}{\theta_1}$$

Problem 5. — Quadrant and Semicircle.

Quadrant.

If $\theta_1 = 90°$, deduce the values of x_0 and y_0.

Semicircle.

If $\theta_1 = 180°$, prove that

$$x_0 = 0, \quad y_0 = \frac{4r}{3\pi}.$$

Problem 6. — Quadrant of the Ellipse.

Let AOB be the quadrant of an ellipse (Fig. 110), whose semi-major and semi-minor axes are $OA = a$ and $OB = b$.

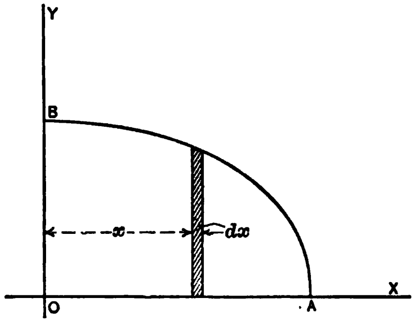

FIG. 110.

Solution. — The equation of the ellipse referred to the axes OX and OY will be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Consider the area to be made up of elementary strips, parallel to OY, of width dx. The area of one of these strips will be

$$dA = y\,dx = \frac{b}{a}\sqrt{a^2 - x^2}\,dx.$$

Hence

$$x_0 = \frac{\int x\,dA}{\int dA} = \frac{\frac{b}{a}\int_0^a x\sqrt{a^2 - x^2}\,dx}{\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx}.$$

$$= \frac{\left[-\frac{1}{3}(a^2 - x^2)^{\frac{3}{2}}\right]_0^a}{\left[\frac{1}{2}x(a^2 - x^2)^{\frac{1}{2}} + \frac{1}{2}a^2\sin^{-1}\frac{x}{a}\right]_0^a} = \frac{4a}{3\pi}.$$

In a similar way we may prove

$$y_0 = \frac{4b}{3\pi}.$$

Problem 7. — Semi-ellipse.

Prove that for the semi-ellipse, bounded by the axis YY,

$$y_0 = 0, \quad x_0 = \frac{4a}{3\pi}.$$

Problem 8. — Plane Areas, which can be divided into Rectangles, or Triangles.

In these cases the coördinates of the centers of gravity may be found by the method stated in Art. 99. For example, let us find the coördinates of the center of gravity of the area (Fig. 111).

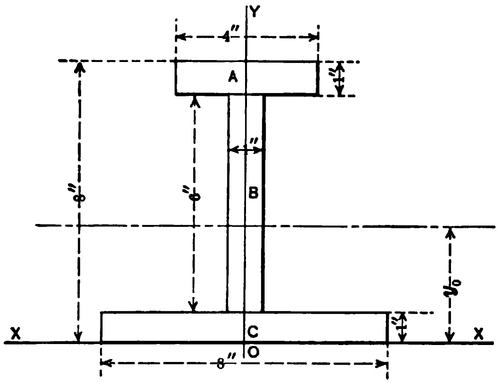

Fig. 111.

Solution. — Divide the section into areas the centers of gravity of which are known; in this case, three rectangles A, B and C. Take moments about OX and we have

$$y_0 = \frac{\Sigma Ay}{A} = \frac{8 \times \frac{1}{2} + 6 \times 4 + 4 \times 7\frac{1}{2}}{8 + 6 + 4} = \frac{58}{18} = \frac{29}{9} = 3.22 \text{ ins.}$$

As the center of gravity will be on the axis of symmetry, it is evident that $x_0 = 0$.

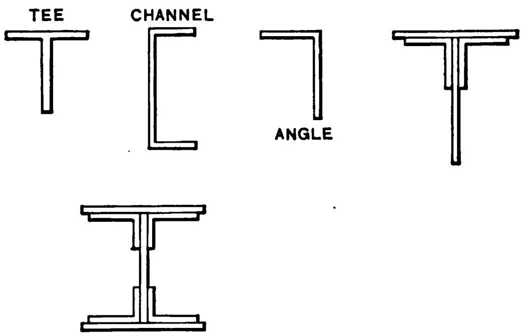

Fig. 112.

This method may be applied in finding the centers of gravity of structural shapes, or built-up sections, as suggested by the above sketches (Fig. 112).

Problem 9. — Trapezoid.

Find the perpendicular distance of the center of gravity of the trapezoid from the larger base.

Solution. — Let the bases of the trapezoid be equal to B and b and its altitude $= h$ (Fig. 113). Divide the area into two triangles, A and C, the distances of whose centers of gravity from OX will be equal to $\frac{h}{3}$ and $\frac{2h}{3}$, respectively.

Then
$$y_0 = \frac{\Sigma Ay}{A} = \frac{\frac{Bh}{2} \times \frac{h}{3} + \frac{bh}{2} \times \frac{2h}{3}}{(B+b)\frac{h}{2}} = \frac{h}{3} \times \frac{B+2b}{B+b}.$$

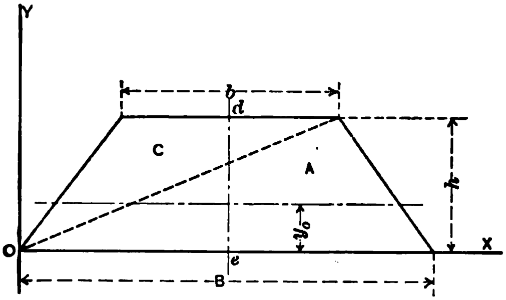

FIG. 113.

By considering the area to be divided into strips parallel to the base B we can show that, since the center of gravity of each strip is its middle point, the center of gravity of the whole area is on the median line de of the trapezoid.

Problem 10. — Circular Half Segment.

By the method of Art. 99 we may deduce the formulas for the center of gravity of the half segment ABC of the circle (Fig. 109). In this case the area of the half segment will be equal to the difference of the areas of the sector AOB and the triangle BOC, the coördinates of the centers of gravity of which have been determined (Probs. 4 and 2). The moment of the sector about OY will be equal to
$$\frac{2}{3} r \frac{\sin\theta_1}{\theta_1} \times \frac{r^2\theta_1}{2} = \frac{r^3}{3}\sin\theta_1,$$

and the moment of the triangle will be
$$\frac{1}{2} r\sin\theta_1\, r\cos\theta_1 \times \frac{2}{3} r\cos\theta_1 = \frac{r^3}{3}\sin\theta_1\cos^2\theta_1.$$

Therefore
$$x_0 = \frac{\Sigma Ax}{A} = \frac{\frac{r^3}{3}\sin\theta_1 - \frac{r^3}{3}\sin\theta_1\cos^2\theta_1}{\frac{r^2\theta_1}{2} - \frac{r^2}{2}\sin\theta_1\cos\theta_1}$$
$$= \frac{2}{3} r \frac{\sin^3\theta_1}{\theta_1 - \sin\theta_1\cos\theta_1}.$$

In a similar manner we may find

$$y_0 = \frac{\Sigma A y}{A} = \frac{\frac{r^3}{3}(1 - \cos\theta_1) - \frac{r^3}{6}\cos\theta_1 \sin^2\theta_1}{\frac{r^2\theta_1}{2} - \frac{r^2}{2}\sin\theta_1\cos\theta_1}$$

$$= \frac{1}{3} r \frac{2 - 3\cos\theta_1 + \cos^3\theta_1}{\theta_1 - \sin\theta_1\cos\theta_1}\,.$$

It is evident that the x coördinate of the center of gravity of the *complete segment* would be the same as for the half segment.

Problem 11.

Find the coördinates of the center of gravity of each of the following sections (Figs. 114a, b, c, d, e, f, g, h).

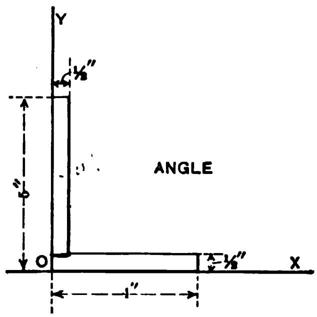

FIG. 114a.

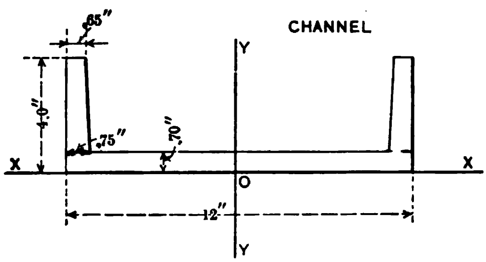

FIG. 114b.

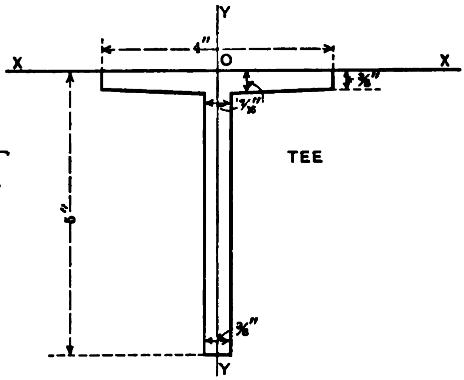

FIG. 114c.

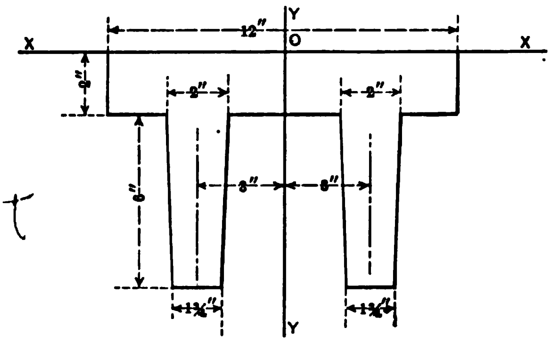

FIG. 114d.

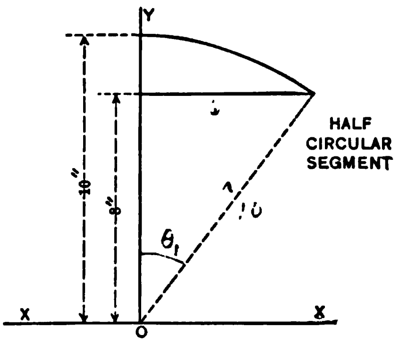

FIG. 114e.

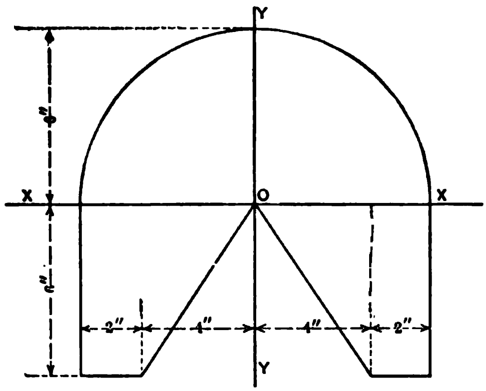

Fig. 114f.

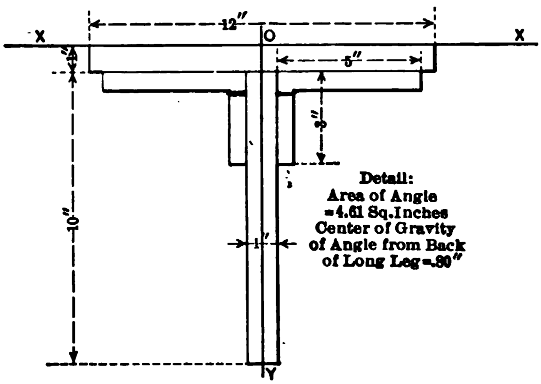

Fig. 114g.

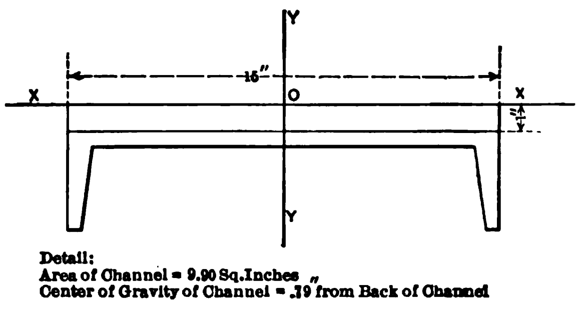

Fig. 114h.

102. Pappus's Theorems.— *Theorem I.— If a plane curve lies wholly on one side of a straight line in its own plane, and, revolving about that line, generates thereby a surface of revolution, the area of the surface is equal to the product of the length of the revolving line, and of the path described by its center of gravity.*

Proof. — Let the curve lie in the Z plane, and let L = its length and x_0 = the distance of its center of gravity from OY.

Then, if the curve is revolved about the axis OY, any elementary length ds, whose distance from OY is equal to x, will describe a surface whose area will be equal to

$$2\pi x\,ds,$$

and the area generated by the entire line will be equal to

$$A = 2\pi \int x\,ds.$$

But $$\int x\,ds = x_0 L \text{ (Art. 93).}$$

Therefore $$A = 2\pi x_0 L. \quad \ldots\ldots\ldots \quad \text{Q.E.D.}$$

It is evident that, if only part of a revolution is made, the area generated will be equal to

$$A_1 = \theta x_0 L,$$

where θ equals the angle through which the plane containing the line is turned.

Theorem II.— If a plane surface, lying wholly on the same side of a straight line in its own plane, revolves about that line, and thereby generates a solid of revolution, the volume of the solid, thus generated, is equal to the product of the revolving area, and of the path described by its center of gravity.

Proof. — Let the surface lie in the Z plane, let A = its area and x_0 = the distance of its center of gravity from OY. Then, if the surface is revolved about the axis OY, any elementary area dA will generate a volume which will be equal to

$$2\pi x\,dA$$

and the volume generated by the whole surface will be equal to

$$V = 2\pi \int x\,dA.$$

But $$\int x\,dA = x_0 A \text{ (Art. 97).}$$

Therefore $$V = 2\pi x_0 A. \quad \ldots\ldots\ldots \quad \text{Q.E.D.}$$

It is evident that if only part of a revolution is made, the angle turned through by the plane of the surface being equal to θ, the volume generated will be equal to $V_1 = \theta x_0 A$.

103. Centers of Gravity of Homogeneous Solids and Systems of Solids. — The general formulas for the coördinates of the center of gravity of the homogeneous solid have been deduced in Art. 90. If the solid can be divided into elementary slices so that the center of gravity of each slice will be on the same straight line, the center of gravity of the entire volume will be somewhere on that line, and it will evidently be necessary to find only one other coördinate.

When a homogeneous solid or system of solids is made up of parts whose centers of gravity and volumes are known, the center of gravity of the whole can evidently be determined by the same method as that given in Art. 91. Thus, if V_1, V_2, V_3, etc., represent the different volumes and (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), etc., their respective centers of gravity, and we let V equal the total volume,

$$V = V_1 + V_2 + V_3 + \text{etc.},$$

and

$$x_0 V = V_1 x_1 + V_2 x_2 + V_3 x_3 + \text{etc.} = \Sigma V x.$$

Hence

$$x_0 = \frac{\Sigma V x}{V} \text{ and similarly } y_0 = \frac{\Sigma V y}{V}, \quad z_0 = \frac{\Sigma V z}{V}.$$

104. Problems. — Centers of Gravity of Homogeneous Solids.

Problem 1. — Cylinder, or Prism, with Bases Parallel.

Find the perpendicular distance from the center of gravity to the plane of the base of a cylinder, or prism. Let h = the altitude and A = the area of the base of the cylinder, or prism (Fig. 115).

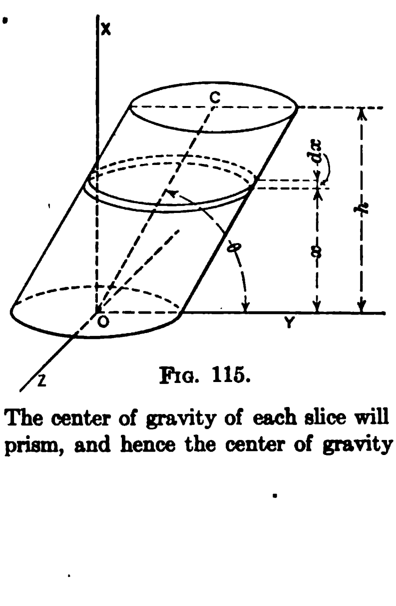

FIG. 115.

Solution. — Assume the coördinate axis OX, perpendicular to the base at its center of gravity, O. Consider the volume to be divided into elementary slices perpendicular to OX and of thickness dx. Then $dV = A\,dx$ and the coördinate of the center of gravity (Art. 90),

$$x_0 = \frac{\int x\,dV}{\int dV} = \frac{A\int_0^h x\,dx}{A\int_0^h dx} = \frac{h}{2}.$$

The center of gravity of each slice will be on the axis OC of the cylinder, or prism, and hence the center of gravity of the entire volume will be on that

axis. If θ is the angle which the axis makes with the base, the distance of the center of gravity from O will evidently be equal to

$$\frac{x_0}{\sin\theta} = \frac{h}{2\sin\theta}.$$

Problem 2. — Cone, or Pyramid.

Find the perpendicular distance of the center of gravity of a cone, or pyramid, from its base. Let c be the center of gravity of the base, A = the area of the base and h = the altitude. (Fig. 116.)

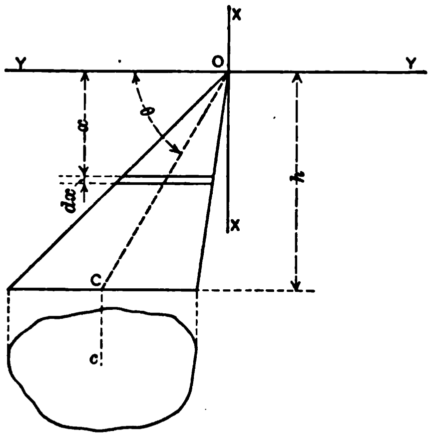

FIG. 116.

Solution. — Refer the cone, or pyramid, to three coördinate axes through the vertex, OX being perpendicular to the base. Consider the volume to be divided into elementary slices perpendicular to OX and of thickness dx.

Then, if a = the area of one of these slices, its volume will be equal to

$$dV = a\,dx;$$

but

$$\frac{a}{A} = \frac{x^2}{h^2}, \quad \text{hence} \quad dV = \frac{A}{h^2}x^2\,dx,$$

and

$$x_0 = \frac{\int x\,dV}{\int dV} = \frac{\frac{A}{h^2}\int_0^h x^3\,dx}{\frac{A}{h^2}\int_0^h x^2\,dx} = \frac{3}{4}h.$$

$$V = \frac{A}{h^2}\int_0^h x^2\,dx = \frac{Ah}{3}.$$

Since the center of gravity of each slice will be on the line OC, the center of gravity of the entire volume will be on that line and its perpendicular distance from the base will be $\frac{h}{4}$. If θ equals the angle which OC makes with the base, the distance of the center of gravity from C will evidently be equal to $\frac{h}{4\sin\theta}$.

By changing the limits of integration we might deduce the formulas for the coördinate of the center of gravity and the volume of the truncated cone.

Problem 3. — Semi-ellipsoid.

The equation of the ellipsoid referred to three coördinate axes through its center is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

where $2\,a$, $2\,b$, $2\,c$ are the major, minor and intermediate axes, respectively. Find the coördinates of the center of gravity of the semi-ellipsoid to the right of the X plane through O (Fig. 117).

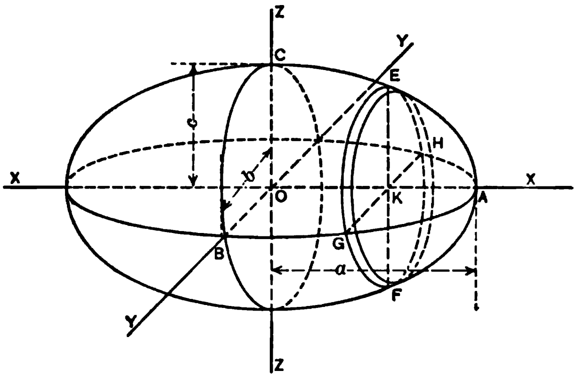

FIG. 117.

Solution. — Consider the volume to be divided into elementary slices perpendicular to OX.

Let $GEHF$ be one of these slices, whose thickness is dx and distance from O is equal to x. The area of this slice will be that of an ellipse, whose semi-major and semi-minor axes, EK and GK, can be found by substituting $y = 0$ and $z = 0$ in the equation for the ellipsoid.

When $\quad y = 0, \quad z = EK = \frac{c}{a}\sqrt{a^2 - x^2},$

and when $\quad z = 0, \quad y = GK = \frac{b}{a}\sqrt{a^2 - x^2}$

Hence the area of $GEHF$ will be equal to

$$\pi\,(GK \times EK) = \frac{\pi bc}{a^2}\,(a^2 - x^2),$$

and the volume of the elementary slice

$$dV = \frac{\pi bc}{a^2}\,(a^2 - x^2)\,dx.$$

Then
$$x_0 = \frac{\int x\,dV}{\int dV} = \frac{\frac{\pi bc}{a^2}\int_0^a (a^2x - x^3)\,dx}{\frac{\pi bc}{a^2}\int_0^a (a^2 - x^2)\,dx} = \frac{3}{8}a. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$V = \frac{\pi bc}{a^2}\int_0^a (a^2 - x^2)\,dx = \frac{2}{3}\pi\,abc. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since the center of gravity of each slice is on OX, the center of gravity of the entire volume will be on OX, and

$$y_0 = 0, \quad z_0 = 0.$$

Problem 4. — Hemisphere.

Find the distance of the center of gravity of the hemisphere from the center of the sphere, the radius of the sphere being equal to r.

Solution. — If, in the case of the ellipsoid (Prob. 3), we let $a = b = c = r$ we have

$$x_0 = \frac{3}{8}r, \quad y_0 = 0, \quad z_0 = 0,$$

and

$$V = \frac{2}{3}\pi r^3.$$

The problem might be solved by integration, directly, without the formulas for the ellipsoid.

Problem 5.

Find the center of gravity of the volume (Fig. 118).

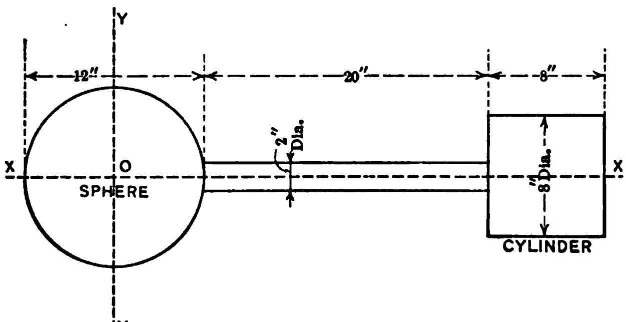

FIG. 118.

This problem may be solved by the method given in Art. 103. Without appreciable error, the rod connecting the cylinder and the sphere may be considered as a cylinder, the curvature of the surface between the rod and the sphere being neglected.

Problem 6.

Find the center of gravity of the volume of revolution (Fig. 119). If the weight of the material per cubic inch is $w = \frac{1}{4}$ pound, find the total weight.

Consider the 2-inch rod as a cylinder, neglecting the curvature of the ends.

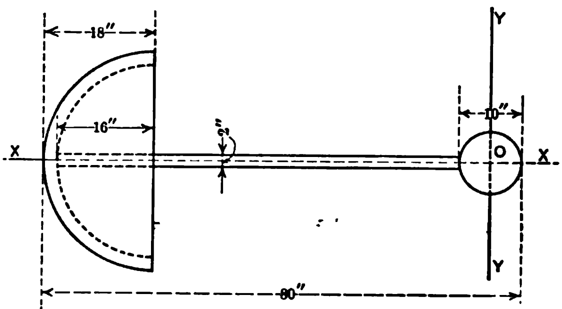

FIG. 119.

Problem 7.

Find the volume and the center of gravity of a truncated right circular cone, whose altitude = 12 inches, diameter of larger base = 10 inches and diameter of smaller base = 6 inches.

Problem 8.

The weight (Fig. 120) is made up of a pyramid with a base 16 inches square, surmounting a truncated pyramid with bases 16 inches square and 20 inches

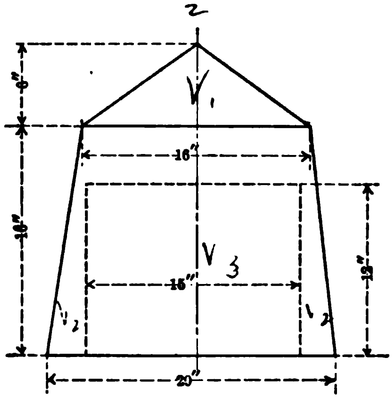

Fig. 120.

square, and has a cylindrical cavity in the center, 15 inches in diameter and 12 inches high. If the weight of the material is 0.26 pound per cubic inch, find the total weight and the center of gravity.

105. Experimental Methods of Determining Centers of Gravity. — Some bodies are of such irregular form that it is impracticable to find their centers of gravity by the methods already explained. It is possible in many of these cases to determine the position of the center of gravity by experiment.

The three following methods are suggested:

(a) *Method of Suspension.* — Suspend the body from some point and in some way mark on the body the position of the vertical line through the point of suspension. Repeat the operation for one or more additional points. Since the center of gravity is on each of the lines so determined, it will be at their point of intersection.

(b) *Method of Balancing.* — Balance the body on a straightedge and in some way mark the position of the vertical plane containing the straightedge; repeat this operation twice, using different balancing positions. Since the center of gravity is in each of these planes it must be at their intersection.

(c) *Method of Weighing.* — Support the body at three points and weigh the supporting forces. The sum of these will equal the entire weight. The moment of any one of the supporting forces, about an axis through the points of application of the other two, will be equal to the moment of the entire body about the axis, and thus the distance from that axis to the vertical through the center of gravity may be computed. By repeating the computation for one of the other supporting forces the vertical through the center of gravity can be located.

By repeating this operation for three other supports, the position of the center of gravity can be found.

If the body is in the form of a straight rod whose center of gravity is on its axis, it is necessary to determine only one coordinate and two supports only will be required, the moment of one supporting force about an axis through the other being equal to the moment of the entire weight about that axis.

CHAPTER IV.

MOMENT OF INERTIA.

106. Moment of Inertia. — The term *moment of inertia* is applied in Engineering to a number of mathematical expressions which represent second moments of areas, weights and masses, with respect to different axes, such as,

$$\int x^2\,dA, \quad \int r^2\,dA, \quad \int r^2\,dw, \quad \int r^2\,dM, \text{ etc.}$$

At present no physical conception of moment of inertia can be given although later it will be shown that the term, as applied to solids, represents a quantity which is proportional to the velocity of rotation produced under certain conditions.

As applied to the area, the moment of inertia is a numerical quantity only, entering into a large number of engineering computations, and takes its name from the analogy between the mathematical expression for it and that for the moment of inertia of a solid.

For the sake of brevity the moments of inertia, $\int x^2\,dA$, $\int r^2\,dw$, etc., will be represented by the symbols I, I_1, I_2, I_x, I_y, etc., the subscripts, where used, indicating the axes with respect to which the moments of inertia are computed.

It is evident from the form of the above expressions that the moment of inertia will always be a positive quantity, differing therein from the moment, which may be either positive or negative.

107. Moment of Inertia of an Area. — The term *moment of inertia* when applied to a plane area may be defined as follows:

The moment of inertia of an area about an axis is the limit of the sum of the products of the elementary areas, into which it may be conceived to be divided, and the squares of their distances from the axis. The axis is called the *inertia axis.*

Let *abc* (Fig. 121) represent any plane area and refer it to three coördinate axes, OX, OY and OZ; OX and OY being in the plane of

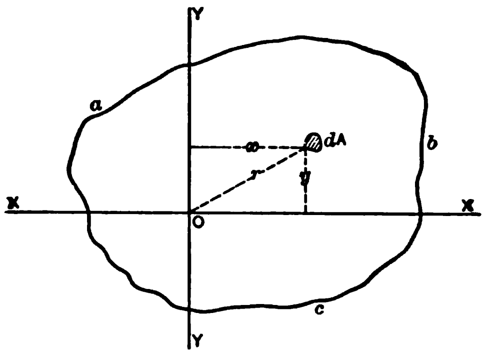

Fig. 121.

the area. Let dA represent an elementary area, whose coördinates are (x, y) and distance from the origin is r; we then shall have

$$I_x = \int y^2\,dA, \qquad I_y = \int x^2\,dA, \qquad I_z = \int r^2\,dA,$$

where I_x, I_y and I_z are the moments of inertia of the area about the axes OX, OY and OZ, respectively.

The moment of inertia, $I_z = \int r^2\,dA$, about the axis OZ, is called the *polar moment of inertia* about that axis. The polar moment of inertia may also be defined as the moment of inertia of an area with respect to a point in its surface.

108. Units of Moments of Inertia of Areas. — Since the moment of inertia of an area is the sum of the products of the elementary areas and the squares of their distances from the inertia axis, it is evident the result will be expressed in the *linear unit to the fourth power*. Thus, if the *inch*, or *foot*, is used as the linear unit, the moment of inertia will be expressed in $(inches)^4$, or $(feet)^4$, which would be read "inches to the fourth power," or "feet to the fourth power," respectively. No shorter names have ever been definitely given to these units, but for the sake of brevity we designate them by such expressions as "inches fourth," "inch units," or simply, "inches," and similar expressions when the dimensions are in feet. The moment of inertia expressed in $(\text{ins.})^4$ will evidently be equal to the moment of inertia expressed in $(\text{ft.})^4$ multiplied by $(12)^4$.

109. Radius of Gyration. — The *radius of gyration of a solid* is the distance from the inertia axis to that point in the solid at which, if its entire mass could be concentrated, its moment of inertia would remain unchanged. Thus if the entire mass of a body, M, is considered to be concentrated at a single point, and ρ is equal to the distance of that point from the inertia axis, the expression for moment of inertia, $\int r^2\,dM$, will be equal to $\rho^2 M$.

Therefore
$$I = \rho^2 M,$$
and the radius of gyration,
$$\rho = \sqrt{\frac{I}{M}}.$$

In the case of the plane area the moment of inertia may, by analogy, be expressed as
$$I = \int x^2\,dA = \rho^2 A$$
where A is equal to the entire area and ρ is its radius of gyration with respect to the inertia axis.

Hence
$$\rho = \sqrt{\frac{I}{A}}.$$

The radius of gyration of both the solid and the plane area will evidently be expressed in linear units.

110. Polar Moment of Inertia. — *Proposition:* — *The polar moment of inertia of an area, with respect to any point in its plane, is equal to the sum of the moments of inertia with respect to any two axes, in the plane of the area and at right angles to each other, passing through that point.*

Proof. — The polar moment of inertia of the area *abc* with respect to the point O (Fig. 121) is equal to
$$I_z = \int r^2\,dA \text{ (Art. 107).}$$

But since
$$r^2 = x^2 + y^2,$$
$$I_z = \int (y^2 + x^2)\,dA = I_x + I_y \quad . \quad . \quad (1) \text{ Q.E.D.}$$

111. Relations between Moments of Inertia of Areas about Parallel Axes. — *Proposition:* — *The moment of inertia of an area about an axis in its plane, not passing through its center of gravity,*

is equal to its moment of inertia about a parallel axis, passing through its center of gravity, increased by the product obtained by multiplying the area by the square of the distance between the two axes.

Proof. — Let abc be any plane surface whose area is equal to A and let YY be any axis in its plane (Fig. 122). Through O_1, the center of gravity, assume the axis Y_1Y_1 parallel and the axis OO_1X perpendicular to YY.

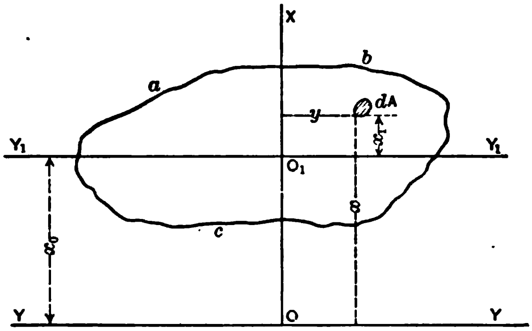

FIG. 122.

Let dA be any elementary area whose coördinates are (x, y) with respect to the axes OY and OX and (x_1, y) with respect to the axes O_1Y_1 and O_1X and let the distance between the axes be equal to x_0. Let $I = \int x^2\,dA$ equal the moment of inertia of the area about the axis YY and $I_0 = \int x_1^2\,dA$ equal the moment of inertia about the axis Y_1Y_1.

Since $\quad x = x_1 + x_0$, we have

$$I = \int x^2\,dA = \int (x_1^2 + 2\,x_1x_0 + x_0^2)\,dA$$

$$= \int x_1^2\,dA + x_0^2 \int dA + 2\,x_0 \int x_1\,dA$$

$$= I_0 + x_0^2A + 2\,x_0 \int x_1\,dA. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The quantity $\int x_1\,dA$, the moment of the area about the axis Y_1Y_1 (Art. 98), is evidently equal to zero, since Y_1Y_1 passes through the center of gravity.

Therefore, $\qquad I = I_0 + x_0^2A. \quad . \quad . \quad . \quad . \quad (2)$ Q.E.D.

It is evident that equation (1) gives the relation between the moments of inertia with respect to any two parallel axes in the plane of the area which may be expressed as follows: The moment of inertia of a plane area about an axis in its plane is equal to its moment of inertia about any parallel axis plus twice the product of its moment about that axis and the distance between the axes plus the product of the area and the square of the distance between the axes.

In a similar manner it can be proved that the polar moment of inertia of the area, with respect to any point O, is equal to its polar moment of inertia with respect to its center of gravity plus the product of the area and the square of the distance between the points (see Art. 133).

Since $I = \rho^2 A$ (Art. 109), if we let ρ_0 equal the radius of gyration about the axis Y_1Y_1, through the center of gravity, and ρ equal that about the parallel axis YY, we shall have by substituting in equation (2)

$$\rho^2 A = \rho_0^2 A + x_0^2 A,$$

and hence

$$\rho^2 = \rho_0^2 + x_0^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which gives us the relation between the radii of gyration of an area about two parallel axes, one of which passes through the center of gravity.

112. Problems. — Deduction of Formulas for Moments of Inertia of Plane Areas.

In deducing the formula for the moment of inertia of a plane area, about an axis in its plane, we may consider the area to be divided into strips of infinitesimal width parallel to the inertia axis. Since every point in one of the strips will be at the same distance from the axis, the moment of inertia of a given strip will be equal to the product of its area and the square of its distance from the axis. Hence, in the formula $I = \int x^2\,dA$, dA may be taken equal to the area of a strip of width dx parallel to and at a distance x from the inertia axis.

Problem 1. — Rectangle or Parallelogram.

Deduce the formulas for the moment of inertia of the rectangle, or parallelogram, about an axis through its base and also about an axis through its center of gravity, parallel to the base (Fig. 123): also for the moment of inertia of

the rectangle about an axis through the center of gravity, perpendicular to the base, and for the polar moment of inertia about its center of gravity.

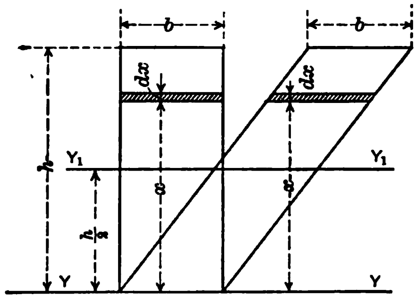

FIG. 123.

Solution.—The following solution will evidently apply to the rectangle or parallelogram. Let b = the base, and h = the altitude, and let I = the moment of inertia about the axis YY through the base, and I_0 = the moment of inertia about the parallel axis Y_1Y_1 through the center of gravity. Consider the rectangle, or parallelogram, to be divided into strips of width dx, parallel to YY. Then $dA = b\,dx$ and the moment of inertia (Art. 107) will equal

$$I = \int x^2\,dA = b\int_0^h x^2\,dx = \frac{bh^3}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The moment of inertia about the axis through the center of gravity may be found by integration of equation (1) between the limits $\frac{h}{2}$ and $-\frac{h}{2}$, or, by the equation $I = I_0 + x_0^2 A$ (Art. 111). Using the latter method,

$$\frac{bh^3}{3} = I_0 + \frac{h^2}{4} \times bh.$$

Therefore,
$$I_0 = \frac{bh^3}{12}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In a similar manner we may show that the moment of inertia of the rectangle about a vertical axis through its center of gravity is equal to

$$\frac{b^3h}{12}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The polar moment of inertia of the rectangle with respect to its center of gravity will be equal to

$$I_z = \frac{bh^3}{12} + \frac{b^3h}{12} = \frac{bh}{12}(b^2 + h^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Problem 2. — Triangle.

Deduce the formulas for the moment of inertia of a triangle (Fig. 124) about the axes:

(a) Through the apex parallel to the base.
(b) Through the center of gravity parallel to the base.
(c) Through the base.

Solution. — Refer the triangle to the rectangular coördinate axes, OX and OY, with the origin at the apex of the triangle and OX perpendicular to the base. Let Y_1Y_1 be the axis through the center of gravity, parallel to the base.

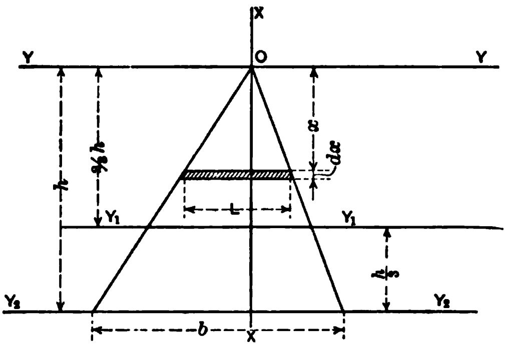

FIG. 124.

Let b = the base, h = the altitude and let I = the moment of inertia about the axis YY, I_0 = the moment of inertia about the axis Y_1Y_1 and I_2 = the moment of inertia about the axis through the base.

(a) Axis through the apex.

Consider the triangle to be divided into elementary strips of width dx, parallel to the base. Then $dA = L\,dx = \frac{b}{h}x\,dx$ and the moment of inertia about the axis YY will be equal to

$$I = \int x^2\,dA = \frac{b}{h}\int_0^h x^3\,dx = \frac{bh^3}{4}. \qquad (1)$$

(b) Axis through the center of gravity.

Substituting in the formula $I = I_0 + x_0^2A$ (Art. 111) we have

$$\frac{bh^3}{4} = I_0 + \frac{4\,h^2}{9} \times \frac{bh}{2}.$$

Therefore $$I_0 = \frac{bh^3}{36}. \qquad (2)$$

(c) Axis through the base.

Substituting in the formula $I = I_0 + x_0^2A$,

$$I_2 = \frac{bh^3}{36} + \frac{h^2}{9} \times \frac{bh}{2} = \frac{bh^3}{12}. \qquad (3)$$

Problem 3. — Circle.

Deduce the formulas for:

(a) The polar moment of inertia of the circle with respect to its center.

(b) The moment of inertia of the circle about a diameter.

Solution. — Let r = the radius of the circle and XX and YY any two diameters at right angles.

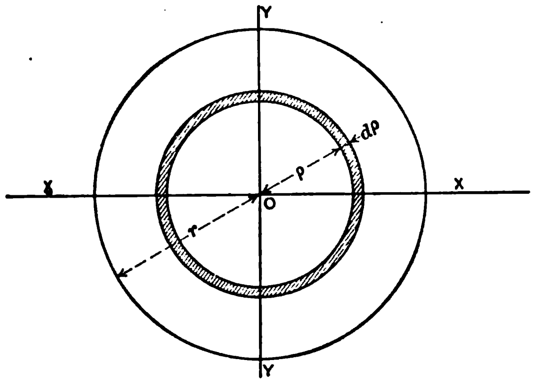

FIG. 125.

(a) Consider the area to be divided into concentric rings, of width $d\rho$, and let the radius of any one of these rings equal ρ. Its area will equal

$$dA = 2\pi\rho\, d\rho$$

and, since every point in the ring is at the same distance from the center, it is evident that its moment of inertia with respect to O will equal

$$r^2\, dA = 2\pi\rho^3\, d\rho.$$

Hence the polar moment of inertia of the circle will equal

$$I_z = \int r^2\, dA = \int_0^r 2\pi\rho^3\, d\rho = \frac{\pi r^4}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It is evident that the polar moment of inertia of a sector, subtending an angle θ at the center, will be equal to

$$I_z = \int_0^r \theta\rho^3\, d\rho = \frac{\theta r^4}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If $\qquad \theta = 90°, \quad I_z = \dfrac{\pi r^4}{8}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$

If $\qquad \theta = 180°, \quad I_z = \dfrac{\pi r^4}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$

(b) The moment of inertia about a diameter may be determined from the polar moment of inertia by the formula (Art. 110),

$$I_z = I_x + I_y.$$

Since for the circle the moments of inertia about all diameters are equal,

$$I_x = I_y = \frac{I_z}{2} = \frac{\pi r^4}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

By similar reasoning the moment of inertia of the circular quadrant about either of the diameters bounding its surface will be equal to

$$I_x = I_y = \frac{\pi r^4}{16}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Since the moment of inertia is always positive (Art. 106), the moments of inertia of the semicircle about its diameter and the diameter at right angles will evidently each be equal to the sum of the moments of inertia of two quadrants. Hence

$$I_x = I_y = \frac{\pi r^4}{8}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Problem 4. — Ellipse.

Deduce the formulas for the moments of inertia of the ellipse about its major and minor axes (Fig. 126), also the formula for the polar moment of

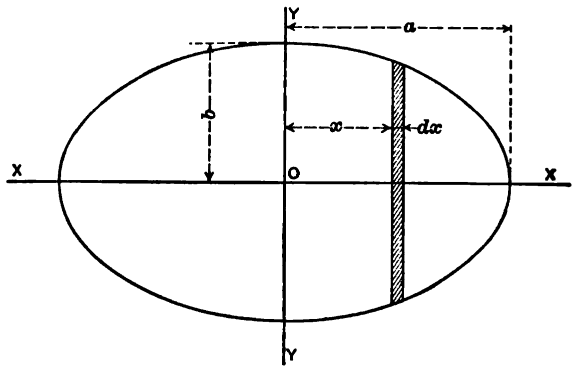

FIG. 126.

inertia with respect to the center. Given the equation of the ellipse, referred to the axes OX and OY, coinciding with the major and minor axes $2a$ and $2b$, respectively, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution. — If we consider the area to be divided into strips parallel to OY, the area of one of these strips will equal

$$dA = 2y\,dx = \frac{2b}{a}(a^2 - x^2)^{\frac{1}{2}}\,dx,$$

and the moment of inertia of the ellipse about the axis YY will be

$$\begin{aligned} I_y &= \int x^2\,dA = \frac{2b}{a}\int_{-a}^{a} x^2 (a^2 - x^2)^{\frac{1}{2}}\,dx \\ &= \frac{2b}{a}\left[\frac{x}{8}(2x^2 - a^2)(a^2 - x^2)^{\frac{1}{2}} + \frac{a^4}{8}\sin^{-1}\frac{x}{a}\right]_{-a}^{a} \\ &= \frac{\pi a^3 b}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1) \end{aligned}$$

In a similar manner by taking strips parallel to OX we may obtain

$$I_x = \frac{\pi a b^3}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

When $a = b$ both I_x and I_y reduce to the form for the moment of inertia of the circle (Prob. 3).

The polar moment of inertia of the ellipse with respect to its center will be equal to

$$I_z = I_x + I_y = \frac{\pi ab}{4}(a^2 + b^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

113. Summary of Formulas for Moments of Inertia of Areas.—It will be found desirable to memorize the following formulas for moments of inertia of areas which have been deduced in the preceding problems.

The formulas are expressed in terms of the dimensions, as determined in the problems (Art. 112), and also in terms of A, the total area in each case. Since $I = A\rho^2$ (Art. 109) the value of ρ, the radius of gyration, may be easily determined for each area.

$\frac{bh^3}{12} = \frac{Ah^2}{12}$ (Rectangle, axis through center of gravity).

$\frac{bh^3}{3} = \frac{Ah^2}{3}$ (Rectangle, axis through base).

$\frac{bh^3}{36} = \frac{Ah^2}{18}$ (Triangle, axis through center of gravity).

$\frac{bh^3}{12} = \frac{Ah^2}{6}$ (Triangle, axis through base).

$\frac{bh^3}{4} = \frac{Ah^2}{2}$ (Triangle, axis through apex).

$\frac{\pi r^4}{4} = \frac{Ar^2}{4}$ (Circle, diameter as axis).

$\frac{\pi a^3 b}{4} = \frac{Aa^2}{4}$ (Ellipse, about axis $2\,b$).

$\frac{\pi ab^3}{4} = \frac{Ab^2}{4}$ (Ellipse, about axis $2\,a$).

Polar moments of inertia with respect to centers of gravity.

$\frac{\pi r^4}{2} = \frac{Ar^2}{2}$ (Circle).

$\frac{\pi ab}{4}(a^2 + b^2) = \frac{A}{4}(a^2 + b^2)$ (Ellipse).

$\frac{bh}{12}(b^2 + h^2) = \frac{A}{12}(b^2 + h^2)$ (Rectangle).

114. Method of Determining the Moments of Inertia of Areas by Dividing into Finite Parts. — Since the moment of inertia is always positive, the moment of inertia of an area about any axis is equal to the sum of the moments of inertia, about that axis, of the parts into which the area may be divided.

In order to use this method the area should be divided into parts, whose moments of inertia about the axis in question can be calculated by using the formulas already deduced. Then, by adding together these values, the moment of inertia of the entire area may be obtained. In certain cases the area may be considered to be the difference of two areas, whence its moment of inertia will be equal to the difference of the moments of inertia of the areas in question.

115. Problems. — Moments of Inertia of Plane Areas. — The following problems may be solved by the method stated in Art. 114, using the formulas given in Art. 113.

Problem 1.

Find the moment of inertia of the section (Fig. 127) about the axis YY, passing through the top of the section, also about axis Y_1Y_1, parallel to YY, through center of gravity.

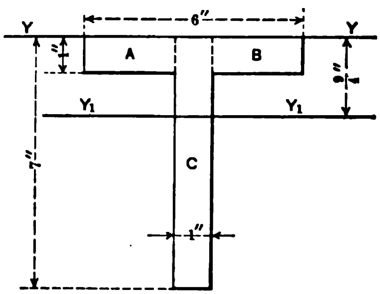

FIG. 127.

Solution. — First find the moment of inertia about the axis YY.

Divide the section into rectangles A, B and C. The moment of inertia of A and B combined will equal

$$\frac{bh^3}{3} = \frac{5 \times 1^3}{3} = \frac{5}{3}.$$

The moment of inertia of C will equal

$$\frac{bh^3}{3} = \frac{1 \times 7^3}{3} = \frac{343}{3}.$$

Hence for the entire area

$$I_y = \frac{5}{3} + \frac{343}{3} = \frac{348}{3} = 116\ (\text{ins.})^4.$$

The distance of the center of gravity from YY will be equal to

$$x_0 = \frac{\Sigma Ax}{A} = \frac{5 \times 0.5 + 7 \times 3.5}{5 + 7} = 2.25\ \text{ins.}$$

Substituting in the formula $I = I_0 + x_0^2 A$

$$116 = I_0 + 12 \cdot (2.25)^2,$$

and the moment of inertia about the axis Y_1Y_1 will be equal to

$$I_0 = 55.25\ (\text{ins.})^4.$$

Problem 2.

Find the moment of inertia of the trapezoid (Fig. 128) about its base.

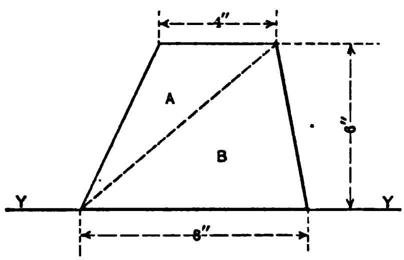

FIG. 128.

Solution. — Divide the area into two triangles, A and B. The moment of inertia of A about the axis YY will equal

$$\frac{bh^3}{4} = \frac{4 \times 6^3}{4} = 216,$$

and the moment of inertia of B about the axis YY will equal

$$\frac{bh^3}{12} = \frac{8 \times 6^3}{12} = 144.$$

Hence $$I_y = 216 + 144 = 360\ (\text{ins.})^4.$$

Problem 3.

Find the moment of inertia of the trapezoid (Fig. 128) about an axis through its center of gravity parallel to its base.

Problem 4.

Find the moment of inertia of the cross section of a hollow circular cylinder about its diameter; outside diameter of cylinder = 12 inches and inside diameter = 8 inches.

Find the polar moment of inertia of the section with respect to its center.

Problem 5.

Find the moments of inertia of the section (Fig. 129) about the axes 1–1 and 2–2 through its center of gravity.

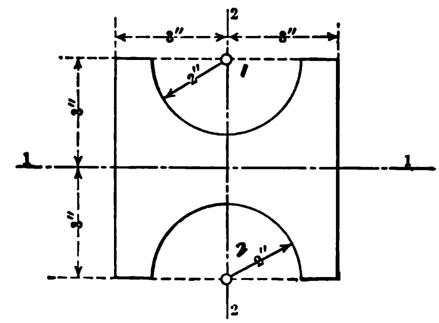

FIG. 129.

116. Moments of Inertia of Structural Shapes. — The moments of inertia of some of the structural shapes can be found by dividing the areas into rectangles and triangles, neglecting the rounds, or fillets, at the corners.

Problems.

Find the moments of inertia of the sections (Figs. 130a, b, c, d, e) about the axes 1-1 and 2-2 passing through their centers of gravity.

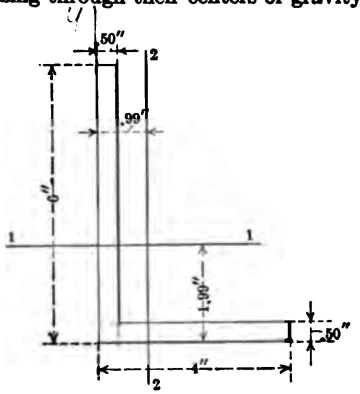

FIG. 130a.

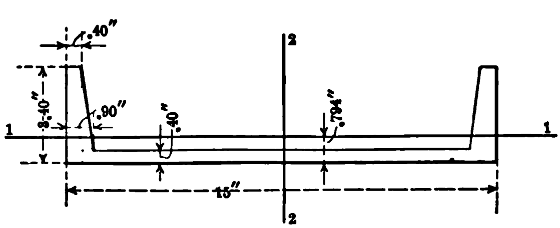

FIG. 130b.

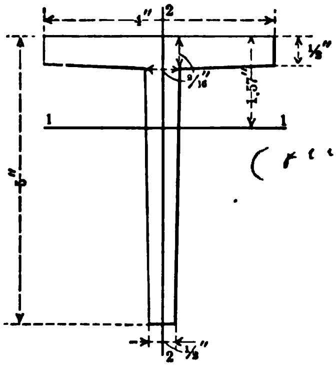

FIG. 130c.

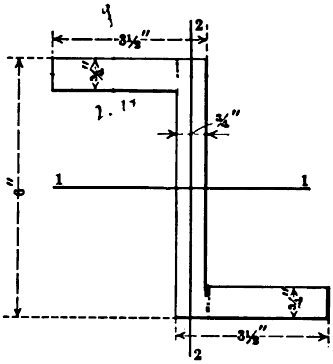

FIG. 130d.

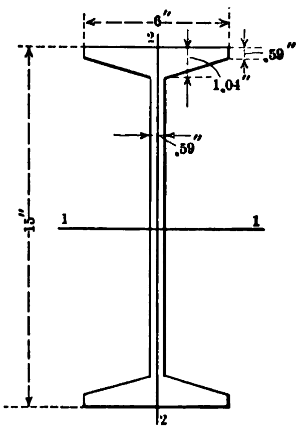

FIG. 130e.

117. Moments of Inertia of Built-up Sections. — In various types of structures (bridges, buildings, etc.), it is frequently necessary to combine structural forms in making up the different parts and, in determining the stresses in these parts, it may be necessary to know the moments of inertia of their cross sections about certain axes passing through their centers of gravity. In the manufacturers' handbooks there are tables in which may be found the distances of the centers of gravity of the sections from the outside surfaces; also, values for the moments of inertia of the sections, about two axes at right angles passing through the center of gravity, similar to the axes 1–1 and 2–2 of forms shown in Fig. 130.

The following problems illustrate the methods of determining moments of inertia of these sections.

Problem 1.

Find the moment of inertia of the section (Fig. 131) about the axis 1–1 through its center of gravity.

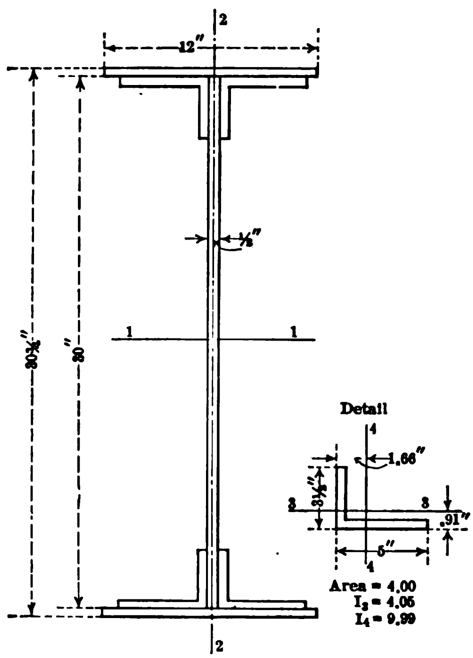

FIG. 131.

The section shown is that of a standard plate girder, made up of a web plate, $30'' \times \frac{1}{2}''$, two flange plates, $12'' \times \frac{3}{8}''$, and four angles, $5'' \times 3\frac{1}{2}'' \times \frac{1}{2}''$,

with the 5″ leg placed horizontally. The area and moments of inertia of the angle are shown in the detail sketch.

Solution. — To determine the moment of inertia I_1, find the difference of the moment of inertia of a rectangle 12″ × 30.75″, including the whole area, and that of a rectangle, 11.5″ × 30″, and add the sum of the moments of inertia of the four angle sections about the axis 1–1. Thus, by use of the formulas (Arts. 111 and 113), we have

$$I_1 = \frac{12 \times (30.75)^3}{12} - \frac{11.5 \times (30)^3}{12} + 4\,[4.05 + 4.00 \times (14.09)^2] = 6392\ (\text{ins.})^4.$$

Problem 2.

Find the moment of inertia of the section (Fig. 131) about the vertical axis 2–2.

Problem 3.

Find the moment of inertia of the section (Fig. 131) if two additional flange plates 12″ × $\frac{3}{8}$″ are added to each of the flanges.

Problem 4.

Find the moments of inertia and the radii of gyration of the built-up sections (Figs. 132a, 132b, 132c), about the axes 1–1 and 2–2, passing through their centers of gravity. Moments of inertia are given in (ins.)^4 and areas in (ins.)^2.

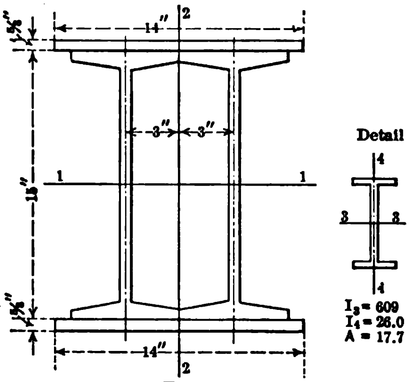

FIG. 132a.

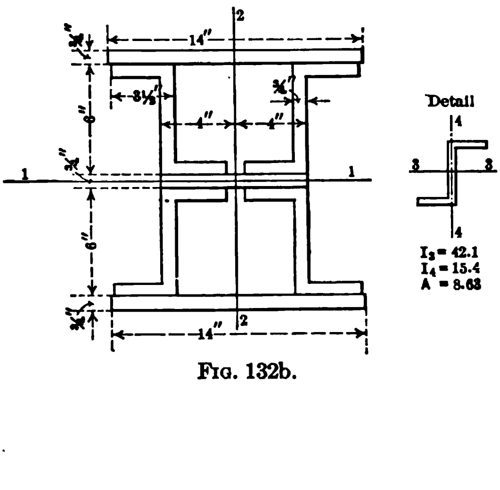

FIG. 132b.

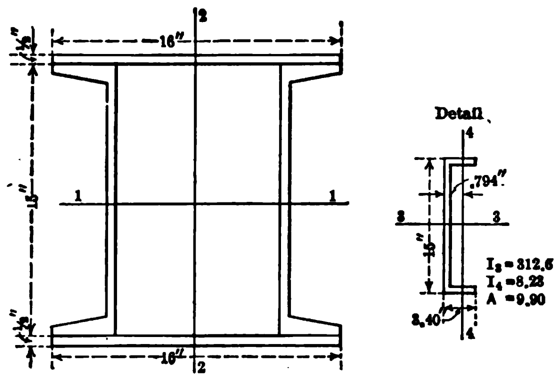

FIG. 132c.

118. Product of Inertia. — There are certain quantities which will occur in the further discussion of moments of inertia, which are second moments, represented by such mathematical expressions as

$$\int xy\,dM, \qquad \int xy\,dW, \qquad \int xy\,dA, \text{ etc.}$$

These quantities are called *products of inertia*, or, sometimes, *moments of deviation*. They are numerical quantities which will be of value only as they will be found to enter into the determination of the relations between moments of inertia of solids and areas with respect to different axes.

We shall use the symbols K, K_1, K_{xy}, etc., to denote products of inertia, the subscripts, when used, indicating the axes, or planes, with respect to which the products are taken.

Unlike the moment of inertia, the product of inertia may evidently be either positive or negative.

119. Product of Inertia of an Area. — *The product of inertia of an area with respect to two coördinate axes may be defined as the limit of the sum of the products of the elementary areas into which it may be conceived to be divided and the product of their distances from the two coördinate axes.*

Referring to Fig. 121, the product of inertia of the area with respect to the axes OX and OY will be

$$K = \int xy\,dA.$$

For that part of the area lying in the first and third quadrants $\int xy\,dA$ will be positive, and for that part in the second and fourth quadrants it will be negative. Whether the product of inertia of the whole area is positive, or negative, will depend on its distribution in the different quadrants; and it is evident that the area might be so located that its product of inertia would be zero.

The units in which the product of inertia of an area is expressed are evidently the same as the units of moments of inertia, the linear dimension entering into the expression for the product of inertia as the fourth power. Hence we may use the symbol (inches)^4, (feet)^4, etc., to denote the units of products of inertia. The product of inertia expressed in (inches)^4 will evidently be $(12)^4$ times its value expressed in (feet)^4.

120. Relation between Products of Inertia of Areas with Respect to Parallel Axes. — *Proposition:* — *The product of inertia of an area with respect to a pair of rectangular coördinate axes is equal to its product of inertia with respect to a pair of parallel axes, passing through its center of gravity, increased by the area multiplied by the product of the coördinates of its center of gravity with respect to the first pair of axes.*

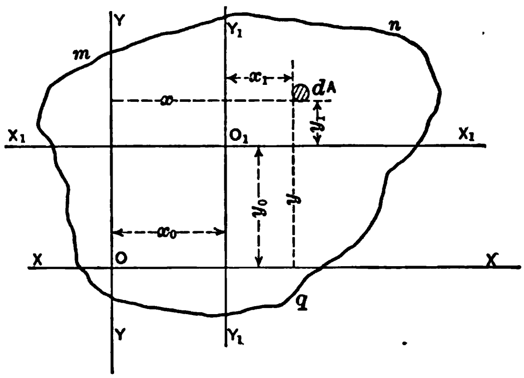

FIG. 133

Proof. — Let A = the area of mnq (Fig. 133) and let OX and OY be any pair of rectangular axes in its plane and O_1X_1 and O_1Y_1 a pair of parallel axes through its center of gravity, O_1.

Let $K = \int xy\,dA$ equal the product of inertia with respect to OX and OY and $K_0 = \int x_1 y_1\,dA$, that for O_1X_1 and O_1Y_1.

Since $x = x_0 + x_1$ and $y = y_0 + y_1$,

$$K = \int xy\,dA = \int (x_1y_1 + x_1y_0 + x_0y_1 + x_0y_0)\,dA$$

$$= \int x_1y_1\,dA + y_0 \int x_1\,dA + x_0 \int y_1\,dA + x_0y_0 \int dA.$$

Since O_1X_1 and O_1Y_1 pass through the center of gravity the statical moments $\int x_1\,dA$ and $\int y_1\,dA$ are evidently equal to zero (Art. 98).

Hence $$K = K_0 + x_0y_0A. \quad . \quad . \quad . \quad (1) \text{ Q.E.D.}$$

121. Problems.— Products of Inertia. To deduce the value of the product of inertia of a plane area with respect to two rectangular coördinate axes in its plane, we may let $dA = dx\,dy$ and determine K from the formula

$$K_{xy} = \int xy\,dA = \int\int xy\,dx\,dy,$$

by double integration.

The expression $\int xy\,dA$ may also be interpreted in the following manner:

If the area is divided into strips parallel to one of the coördinate axes we may let dA equal the area of one of the strips and $xy\,dA$ its product of inertia; in which case x and y will be the coördinates of the center of the strip with respect to the axes. Thus, if the strip is parallel to the X-axis, y will be constant for all points in the strip, and x will be the distance of its center from the Y-axis. Hence we may say that $x\,dA$ is the moment of the area of the strip about the Y-axis and $xy\,dA$ is its product of inertia with respect to the X- and Y-axes. The product of inertia of the entire area will then be equal to

$$\int xy\,dA.$$

This method will be used in the following problems.

Problem 1. — Rectangle.

Find the product of inertia of the rectangle with respect to a pair of axes, OX and OY, coinciding with two of its sides, whose lengths are equal to b and h (Fig. 134).

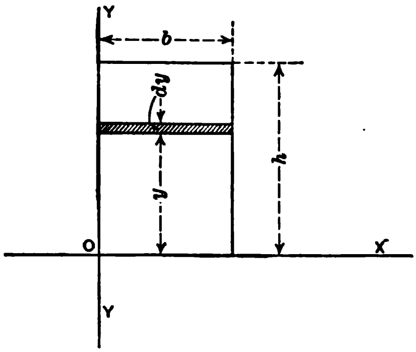

FIG. 134.

Solution. — Consider the area to be divided into elementary strips parallel to OX. Then, for one of these strips, the area $dA = b\,dy$; the moment $x\,dA = \frac{b}{2}b\,dy = \frac{b^2}{2}dy$ (Art. 98); and the product of inertia $xy\,dA = \frac{b^2y}{2}dy$ (Art. 119). Therefore, the product of inertia of the entire area with respect to the axes OX and OY will be equal to

$$K = \frac{b^2}{2}\int_0^h y\,dy = \frac{b^2h^2}{4}. \quad (1)$$

It is evident that if either one, or both, of the axes had been taken through the center of gravity of the rectangle and parallel to the axes given, the value of K would have been zero.

Problem 2. — Triangle.

Find the product of inertia of the right triangle about a pair of axes, OX and OY, coinciding with the sides of the right angle (Fig. 135); also, about a pair of axes through the center of gravity, parallel to OX and OY.

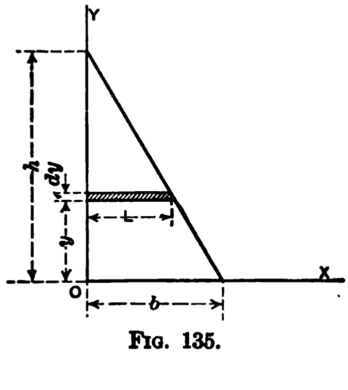

FIG. 135.

Solution. — Let b = the base and h = the altitude of the triangle. Consider the area to be divided into elementary strips parallel to OX.

For one of these strips, the area

$$dA = L\,dy = \frac{b}{h}(h - y)\,dy,$$

the moment $$x\,dA = \frac{b^2}{2h^2}(h - y)^2\,dy \text{ (Art. 98)},$$

and the product of inertia

$$xy\,dA = \frac{b^2}{2h^2}(h - y)^2 y\,dy \quad \text{(Art. 119)}.$$

Therefore the product of inertia of the entire triangle with respect to the axes OX and OY will be equal to

$$K = \int xy\,dA = \frac{b^2}{2h^2}\int_0^h (h^2y - 2hy^2 + y^3)\,dy = \frac{b^2h^2}{24}. \quad . \quad . \quad . \quad (1)$$

By substituting in formula (1) (Art. 120) we may determine the product of inertia of the triangle with respect to a pair of axes parallel to OX and OY through its center of gravity, as follows:

$$\frac{b^2h^2}{24} = K_0 + \frac{b}{3}\times\frac{h}{3}\times\frac{bh}{2}$$

and $$K_0 = -\frac{b^2h^2}{72}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Problem 3. — Circular Quadrant.

Find the product of inertia of the circular quadrant with respect to the radii bounding its area.

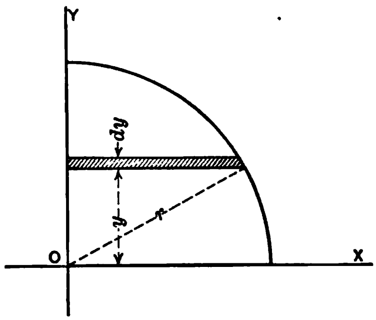

FIG. 136.

Solution. — Let r = the radius of the circle (Fig. 136). Consider the quadrant to be divided into elementary strips parallel to OX. For one of these strips, the area

$$dA = (r^2 - y^2)^{\frac{1}{2}}\,dy,$$

the moment $$x\,dA = \frac{1}{2}(r^2 - y^2)\,dy,$$

and the product of inertia

$$xy\,dA = \frac{1}{2}(r^2 - y^2)\,y\,dy.$$

Therefore for the entire area

$$K = \int xy\,dA = \frac{1}{2}\int_0^r (r^2y - y^3)\,dy = \frac{r^4}{8}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

To determine the product of inertia with respect to a pair of parallel axes through the center of gravity we have

$$x_0 = y_0 = \frac{4\,r}{3\,\pi} \quad \text{(Prob. 5, Art. 201).}$$

Hence

$$\frac{r^4}{8} = K_0 + \left(\frac{4\,r}{3\,\pi}\right)^2 \times \frac{\pi r^2}{4},$$

and

$$K_0 = r^4\left(\frac{1}{8} - \frac{4}{9\,\pi}\right) = -0.0165\,r^4. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

122. Product of Inertia of an Area with Respect to a Pair of Axes, one of which is an Axis of Symmetry. —

Proposition: — The product of inertia of an area with respect to a pair of axes, one of which is an axis of symmetry, is equal to zero.

Proof. — Let abc (Fig. 137) represent any area that is symmetrical with respect to the axis OY. Consider the area to be divided into elementary strips, parallel to the axis OX.

Following the method of Art. 121, the area of any one of these strips will be equal to $dA = L\,dy$ and, since its center is on OY, its moment $x\;dA = 0$ (Art. 98). Hence its product of inertia $xy\,dA = 0$ and therefore the product of inertia of the entire area

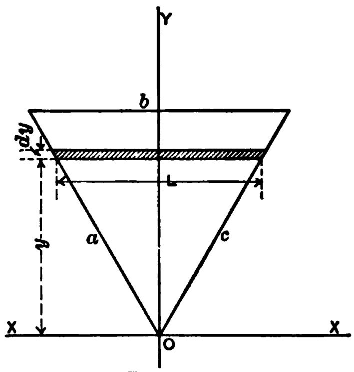

FIG. 137.

$$K = \int xy\,dA = 0.$$

123. Problems. — Illustrating the Method of Determining the Products of Inertia of Areas by Dividing into Finite Parts. — The product of inertia of an area with respect to a pair of rectangular coördinate axes will evidently be equal to the algebraic sum of the products of inertia of the parts, into which it may be divided, with respect to the axes.

Problem 1.

Find the product of inertia of the angle section (Fig. 138) with respect to the axes O_1X_1 and O_1Y_1, through its center of gravity, the coördinates of

which with respect to the center lines, OX and OY, of the legs are shown in the figure.

Solution. — Divide the section into the rectangles M and N as shown. The values of K for both of these rectangles with respect to the axes OX and

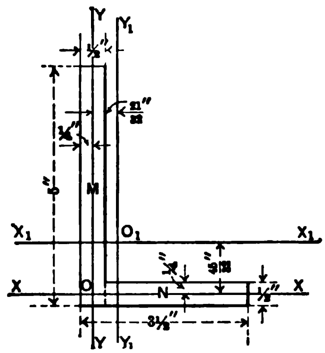

FIG. 138.

OY will equal zero (Art. 122). The combined area of M and N will equal $\frac{5}{2}+\frac{3}{2} = 4$ square inches. Then, by substituting in the formula,

$$K = K_0 + x_0y_0A \text{ (Art. 120)},$$

we have
$$0 = K_0 + \frac{21}{32} \times \frac{45}{32} \times 4,$$

and
$$K_0 = -3.69 \text{ (ins.)}^4.$$

Problem 2.

Find the product of inertia of the trapezoid (Fig. 139) with respect to the parallel sets of coördinate axes, OX and OY; O_2X and O_2Y_2; O_1X_1 and O_1Y_1

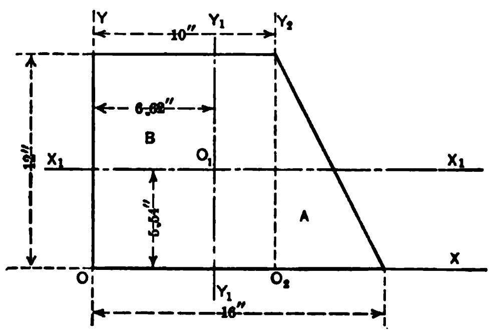

FIG. 139.

through the center of gravity; the coördinates of the center of gravity being shown in the figure.

Solution. — We will first compute K_2, the product of inertia with respect to O_2X and O_2Y_2.

Divide the area into the triangle A and the rectangle B. From formula (2) (Art. 121) we have for the triangle

$$\frac{b^2h^2}{24} = \frac{36 \times 144}{24} = 216,$$

and from formula (1) (Art. 121) we have

$$-\frac{b^2h^2}{4} = \frac{100 \times 144}{4} = -3600,$$

the product of inertia being negative since the area is in the second quadrant. Therefore $K_2 = 216 - 3600 = -3384$ (ins.)^4.

The product of inertia K_0, with respect to the axes through the center of gravity, can next be found by substituting in formula (1) (Art. 120), noting that $x_0 = -3.38$, and $y_0 = 5.54$.

Then $-3384 = K_0 - 3.38 \times 5.54 \times 156,$

and $K_0 = -463$ (ins.)^4.

By substituting again in formula (1) (Art. 120) we may find the value of K, the product of inertia with respect to OX and OY. In this case both x_0 and y_0 are positive and

$$K = -463 + 6.62 \times 5.54 \times 156 = 5258 \text{ (ins.)}^4.$$

124. Relations between the Moments and Products of Inertia of an Area about Different Pairs of Coördinate Axes, Passing through the Same Point. — Let I_x and I_y equal the moments of inertia and K equal the product of inertia of the area abc (Fig. 140) with respect to the rectangular coördinate axes, OX and

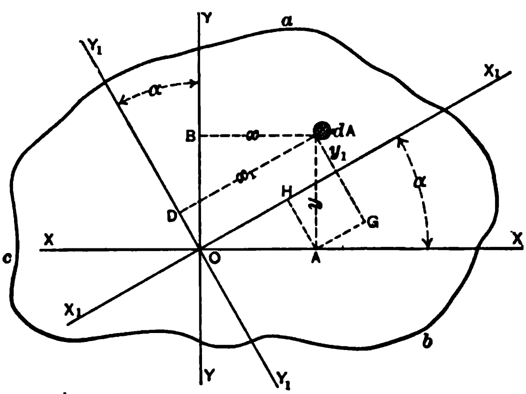

FIG. 140.

OY, through any point O in its surface; and let I_{x_1}, I_{y_1} and K_1 equal the moments and product of inertia with respect to any other pair of rectangular axes, OX_1 and OY_1, passing through this point.

Let dA be any elementary area whose coördinates with respect to OX and OY are (x, y) and with respect to OX_1 and OY_1 are (x_1, y_1).

Let α = the angle between the axes OX and OX_1, and hence between OY and OY_1.

By definition we have

$$I_x = \int y^2\,dA, \qquad I_y = \int x^2\,dA, \qquad K = \int xy\,dA,$$

$$I_{x_1} = \int y_1^2\,dA, \qquad I_{y_1} = \int x_1^2\,dA, \qquad K_1 = \int x_1y_1\,dA.$$

But $\quad x_1 = x\cos\alpha + y\sin\alpha \quad$ and $\quad y_1 = y\cos\alpha - x\sin\alpha,$

and hence

$$\begin{aligned} x_1^2 &= x^2\cos^2\alpha + y^2\sin^2\alpha + 2\,xy\cos\alpha\sin\alpha, \\ y_1^2 &= x^2\sin^2\alpha + y^2\cos^2\alpha - 2\,xy\cos\alpha\sin\alpha, \\ x_1y_1 &= xy\,(\cos^2\alpha - \sin^2\alpha) - (x^2 - y^2)\cos\alpha\sin\alpha. \end{aligned}$$

Substituting these values in the equations for I_{x_1}, I_{y_1} and K_1, we have

$$I_{x_1} = \sin^2\alpha\int x^2\,dA + \cos^2\alpha\int y^2\,dA - 2\cos\alpha\sin\alpha\int xy\,dA,$$

$$I_{y_1} = \cos^2\alpha\int x^2\,dA + \sin^2\alpha\int y^2\,dA + 2\cos\alpha\sin\alpha\int xy\,dA,$$

$$K_1 = \cos\alpha\sin\alpha\left[\int y^2\,dA - \int x^2\,dA\right] + \left[\cos^2\alpha - \sin^2\alpha\right]\int xy\,dA.$$

Therefore

$$I_{x_1} = I_y\sin^2\alpha + I_x\cos^2\alpha - 2\,K\cos\alpha\sin\alpha, \quad . \quad . \quad (1)$$

$$I_{y_1} = I_y\cos^2\alpha + I_x\sin^2\alpha + 2\,K\cos\alpha\sin\alpha, \quad . \quad . \quad (2)$$

$$K_1 = \cos\alpha\sin\alpha\,(I_x - I_y) + (\cos^2\alpha - \sin^2\alpha)\,K. \quad . \quad (3)$$

Equations (1), (2) and (3) express the relations between the moments and products of inertia of a plane area with respect to any two pairs of rectangular axes, passing through any point in its surface.

125. Principal Moments of Inertia of Areas and Principal Axes. — *Proposition:* — *In every area, a given point being assumed as origin, there is at least one pair of rectangular axes, about one of which the moment of inertia is a maximum, and the other a minimum; these moments of inertia being called principal moments*

of inertia, and the axes about which they are taken being called principal axes.

Proof. — By differentiating the value of I_{x_1} (equation 1, Art. 124) and placing the derivative equal to zero we have

$$\frac{dI_{x_1}}{d\alpha} = 2\cos\alpha\sin\alpha\, I_y - 2\cos\alpha\sin\alpha\, I_x - 2K(\cos^2\alpha - \sin^2\alpha) = 0. \quad (1)$$

Solving this equation,

$$\frac{2\cos\alpha\sin\alpha}{\cos^2\alpha - \sin^2\alpha} = \frac{2K}{I_y - I_x},$$

and

$$\tan 2\alpha = \frac{2K}{I_y - I_x}. \quad \ldots \ldots \quad (2)$$

Hence, for the value of α given by equation (2), I_{x_1} will be a maximum, or minimum; and, as there are two values of 2α corresponding to the same value of $\tan 2\alpha$, these two values differing by 180°, the equation will give two values of α, differing by 90°. For one of these values, I_{x_1} will be a maximum and the other a minimum. Q.E.D.

Comparing equation (1) with equation (3) (Art. 124) we have

$$\frac{dI_{x_1}}{d\alpha} = 2K_1,$$

and hence the product of inertia of an area with respect to a pair of principal axes is equal to zero.

It should be noted that by differentiating the value of I_{y_1} (equation 2, Art. 124) and placing the derivative equal to zero we would obtain the same results.

126. Axes of Symmetry of Areas are Principal Axes. — Since for any pair of axes in the plane of an area, one of which is an axis of symmetry, $K = 0$ (Art. 122), it follows that an axis of symmetry is a principal axis (Art. 125). The second principal axis may be any line in the plane of the area perpendicular to the axis of symmetry. Whether the moment of inertia of an area about an axis of symmetry is a maximum or a minimum will depend on the shape of the area and also upon the position of the second principal axis.

When the moments of inertia about both principal axes are equal, it follows that the moments of inertia about all axes, passing through their point of intersection, are equal. Examples of such areas would be the square, circle, regular polygons, certain built-up sections of columns, etc.

127. Ellipse of Inertia. — If OX and OY are principal axes for any plane area (Fig. 141), the value of the moment of inertia about any axis OX_1, making an angle α with OX, may be found by substituting $K = 0$ in equation (1) (Art. 124), which will give

$$I_{x_1} = I_y \sin^2 \alpha + I_x \cos^2 \alpha. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In a similar manner the moment of inertia about the axis OY_1, perpendicular to OX_1, will be equal to

$$I_{y_1} = I_y \cos^2 \alpha + I_x \sin^2 \alpha. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

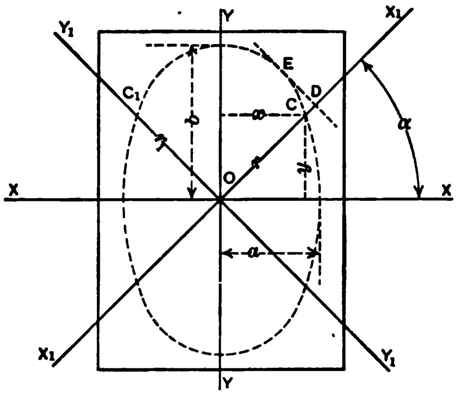

Fig. 141.

If we construct an ellipse with the semi-axes, a and b, coinciding with OX and OY and let r = the length of the semi-diameter OC, the coördinates of C being (x, y), we will have

$$x = r \cos \alpha, \qquad y = r \sin \alpha.$$

Substituting these values in the equation for the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

will give the polar equation

$$a^2 \sin^2 \alpha + b^2 \cos^2 \alpha = \frac{a^2 b^2}{r^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Comparing equations (3) and (1); if we let

$$I_x = b^2 \quad \text{and} \quad I_y = a^2, \text{ we will have}$$

$$I_{x_1} = \frac{a^2 b^2}{r^2} = \frac{I_x I_y}{r^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In a similar manner we can prove that

$$I_{y_1} = \frac{I_x I_y}{r_1^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where r_1 equals the semi-diameter OC_1.

Therefore, if for any area an ellipse is constructed with its semi-major and semi-minor axes equal to the square roots of the principal moments of inertia, the axes of the ellipse being laid off perpendicular to the principal axes, the moment of inertia about any other axis passing through the origin will be equal to the product of the principal moments of inertia, divided by the square of the semi-diameter intercepted on that axis by the ellipse.

The ellipse so constructed is called an *ellipse of inertia, or momental ellipse.*

The proposition shows that the moments of inertia of a plane area about a system of axes passing through any point in its surface are inversely proportional to the squares of the distances intercepted on those axes by the ellipse of inertia, and hence we may draw the following conclusions:

(a) *If the principal moments of inertia are equal, the ellipse of inertia becomes a circle and the moments of inertia about all axes passing through the origin are equal.*

(b) *If the moments of inertia about more than two axes, not rectangular, passing through a given point are equal, the ellipse of inertia must be a circle and the moment of inertia about all axes passing through that point will be the same.*

(c) *For every axis of inertia passing through a given point O, there is another axis, making an equal and opposite angle with the principal axis, about which the moment of inertia is the same.*

By adding equations (1) and (2) we get

$$I_{x_1} + I_{y_1} = I_x + I_y; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

that is, *the sum of the moments of inertia about any pair of rectangular axes passing through a point is equal to the sum of the moments of inertia about the principal axes passing through that point.* This relation would follow directly from the proposition in Art. 110. It will also be evident that the following relation will exist between the radii of gyration

$$\rho_{x_1}^2 + \rho_{y_1}^2 = \rho_x^2 + \rho_y^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Another way of stating the proposition in regard to the ellipse of inertia is the following:

Equation (1) may be written

$$\rho_{x_1}{}^2 = \rho_y{}^2 \sin^2 \alpha + \rho_x{}^2 \cos^2 \alpha. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If ED is the tangent to the eliipse, parallel to OY_1, we have for one of the properties of the ellipse

$$\overline{OD}^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Comparing equations (8) and (9); if we let $\rho_x = a$ and $\rho_y = b$, we shall have $\rho_{x_1} = OD$: which shows that if the principal radii of gyration are laid off along the principal axes, OX and OY, and an ellipse is constructed with these as semi-major and semi-minor axes, the radius of gyration about any other axis, OX_1, will be equal to the distance intercepted on that axis between the origin and the tangent to the ellipse which is perpendicular to OX_1.

128. Problems. — Moments of Inertia about Inclined Axes.

Problem 1.

Find the moment of inertia of the triangle (Fig. 142) about the axis X_2X_2, making an angle of 60° with the principal axis OX through its center of gravity, and intersecting OX at the distance of 10″ from the center of gravity.

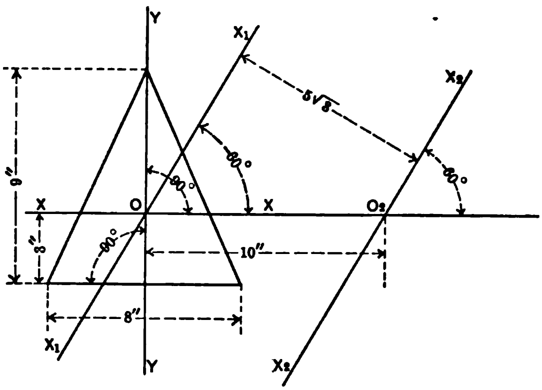

Fig. 142.

Solution. — The moment of inertia about the axis X_1X_1 passing through the center of gravity, parallel to X_2X_2, may be found from the equation

$$I_{x_1} = I_y \sin^2 \alpha + I_x \cos^2 \alpha \text{ (Art. 127).}$$

In this case $\alpha = 60°,\ I_x = 162,\ I_y = 96.$

Substituting these values we have

$$I_{x_1} = 96 \times \frac{3}{4} + 162 \times \frac{1}{4} = 112.5 \text{ (ins.)}^4.$$

Then $\quad I_{x_2} = I_{x_1} + x_0^2 A$ (Art. 111)

and $\quad I_{x_2} = 112.5 + 36 \times (5\sqrt{3})^2 = 2812.5 \text{ (ins.)}^4.$

Problem 2.

Find the principal axes and principal moments of inertia of the section (Fig. 143), the following values being given:

$$I_x = 10.00 \text{ (ins.)}^4,$$
$$I_y = 4.05 \text{ (ins.)}^4,$$
$$K = -3.69 \text{ (ins.)}^4 \text{ (Prob. 1, Art. 123).}$$

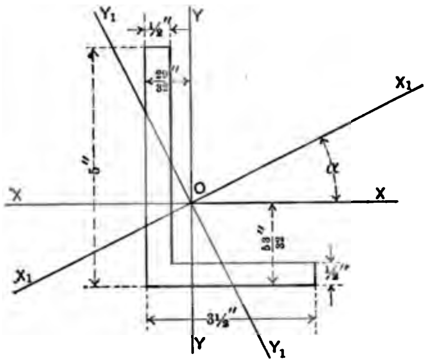

FIG. 143.

Solution. — Substituting in equation (2) (Art. 125) we have

$$\tan 2\alpha = \frac{2K}{I_y - I_x} = \frac{-7.38}{-5.95} = 1.240,$$
$$2\alpha = 51°\,7', \quad \text{or} \quad 231°\,7',$$
$$\alpha = 25°\,34', \quad \text{or} \quad 115°\,34'.$$

Then, from the equation,

$$I_{x_1} = I_y \sin^2\alpha + I_x \cos^2\alpha - 2K\cos\alpha\sin\alpha \text{ (Art. 124),}$$

we have
$$I_{x_1} = 4.05\sin^2(25°\,34') + 10.00\cos^2(25°\,34') + 7.38\,(\cos 25°\,34')\,(\sin 25°\,34')$$
$$= 11.77 \text{ (ins.)}^4 \quad \text{(Maximum),}$$

and
$$I_{y_1} = I_y\cos^2\alpha + I_x\sin^2\alpha + 2K\cos\alpha\sin\alpha$$
$$= 4.05\cos^2(25°\,34') + 10.00\sin^2(25°\,34') - 7.38\,(\cos 25°\,34')\,(\sin 25°\,34')$$
$$= 2.29 \text{ (ins.)}^4 \quad \text{(Minimum).}$$

Note. — If the functions of 115° 34′ were substituted in above equations I_{x_1} would be the moment of inertia about the axis marked OY_1 and I_{y_1} would be the moment of inertia about the axis marked OX_1.

An inspection of the equations will show that it is immaterial which one of the angles is used.

129. Moment of Inertia of a Solid. — The moment of inertia of a solid body with respect to a given axis may be defined as the limit of the sum of the products of the weights of the elementary particles, which make up the body, and the squares of their distances from the axis. If dW denotes the weight of an elementary particle and r its distance from the axis, the moment of inertia with respect to the axis may be expressed as

$$I = \int r^2 \, dW.$$

The moment of inertia may be similarly defined in terms of the mass of the body, in which case

$$I_m = \int r^2 \, dM$$

where dM represents the mass of an elementary particle and r its distance from the axis.

Since $$dM = \frac{dW}{g} \text{ it follows that}$$

$$I_m = \frac{I}{g}.$$

Since we are using the gravitation system of units, we shall always calculate moments of inertia in terms of units of weight, unless otherwise designated.

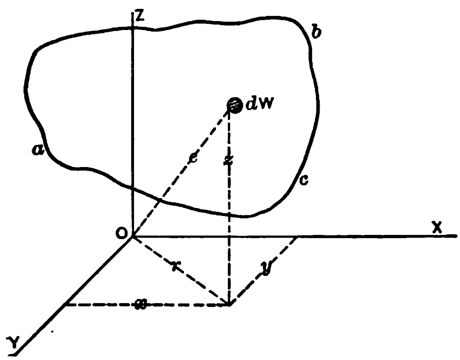

Fig. 144.

In the discussion of moments of inertia of solids we have to deal with the second moments given by the expressions:

$$\int x^2 \, dW, \quad \int y^2 \, dW, \text{ and } \int z^2 \, dW,$$

which may be defined as the moments of inertia with respect to the X, Y and Z planes, respectively, dw being the weight of an elementary particle whose coördinates are (x, y, z) (Fig. 144).

We may also define the quantity $\int e^2\,dW$ as the moment of inertia of the solid with respect to the point O.

An inspection of Fig. 144 will show that the moments of inertia of the solid about the axes OX, OY and OZ may be expressed in terms of the moments of inertia with respect to the planes as follows:

$$I_z = \int r^2\,dW = \int (x^2 + y^2)\,dW = \int x^2\,dW + \int y^2\,dW,$$

$$I_x = \int y^2\,dW + \int z^2\,dW,$$

$$I_y = \int x^2\,dW + \int z^2\,dW.$$

The moment of inertia with respect to the origin, O, will be equal to

$$\int e^2 dW = \int (x^2 + y^2 + z^2)\,dW = \int x^2\,dW + \int y^2 d\,W + \int z^2\,dW.$$

In engineering computations involving moments of inertia of solids, the only values used are those of the moments of inertia with respect to the axes.

130. Units of Moments of Inertia of Solids. — In computing moments of inertia of solids, if we use pounds for units of weight and feet for units of distance, the quantity,

$$I = \int r^2\,dW,$$

will evidently be expressed as the product of pounds and (feet)^2 and we may designate the units as pounds (feet)^2. In a similar manner, if inches and pounds are used in the computation, the units will be pounds (inches)^2. Other units might be tons (feet)^2, kilograms (centimeters)^2, etc. No simple names have been devised for these units, which would be read as "pounds feet squared," "pounds inches squared," etc. We frequently designate them, however, by such terms as foot-pound units, inch-pound units, etc. The moment of inertia expressed in pounds (feet)^2 will evidently be equal to the moment of inertia expressed in pounds (inches)^2 divided by $(12)^2$.

131. Radius of Gyration of a Solid. — The radius of gyration in terms of the mass of a solid has already been defined (Art. 109) as being equal to

$$\rho = \sqrt{\frac{I_m}{M}},$$

where I_m is the moment of inertia in terms of the mass and M is the entire mass of the solid.

Multiplying and dividing by $\sqrt{g}$, we have

$$\rho = \sqrt{\frac{I}{W}},$$

where I equals the moment of inertia in terms of the weight and W equals the entire weight of the solid.

Hence $$I = W\rho^2,$$

that is, the moment of inertia of a solid in terms of units of weight is equal to the product of the weight of the solid and the square of its radius of gyration.

132. Relation between the Moment of Inertia of a Thin Flat Plate and that of an Area. — Let *abc* (Fig. 145) be a homogeneous

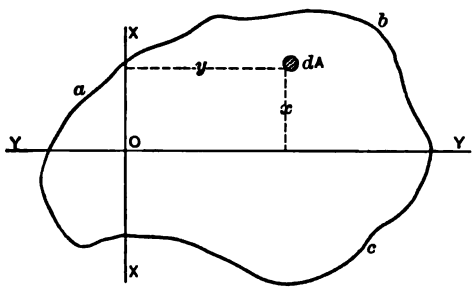

FIG. 145.

thin flat plate of uniform thickness, t, and uniform weight per unit volume, w. Let $I = \int r^2 \, dW$ equal the moment of inertia about the axis YY in the middle layer of the plate. Let the thickness be so small that we may take for the elementary particle a prism, the area of whose base equals dA, whose height equals (t), and perpendicular distance from YY equals x. Then the weight of the particle $dW = wt \, dA$, and its moment of inertia about

YY will equal $wtx^2\,dA$. Hence the moment of inertia of the entire plate about YY will be equal to

$$I = \int r^2\,dW = wt\int x^2\,dA.$$

But $\int x^2\,dA$ is the moment of inertia of the area of the plate about YY.

Therefore, *the moment of inertia of a thin flat plate, of uniform thickness and material, about an axis in its plane is equal to the moment of inertia of the area of the plate about that axis multiplied by the weight per unit volume times the thickness of the plate.*

It is evident that the same relation will exist for an axis perpendicular to the plate.

133. Relations between Moments of Inertia of Solids about Parallel Axes. — *Proposition:* — *The moment of inertia of a solid about any axis is equal to its moment of inertia about a parallel axis, passing through its center of gravity, plus the product of the weight of the solid and the square of the perpendicular distance between the axes.*

Proof. — Let *abc* be any solid, whose weight is equal to W, and let I equal its moment of inertia about any axis ZZ (Fig. 146).

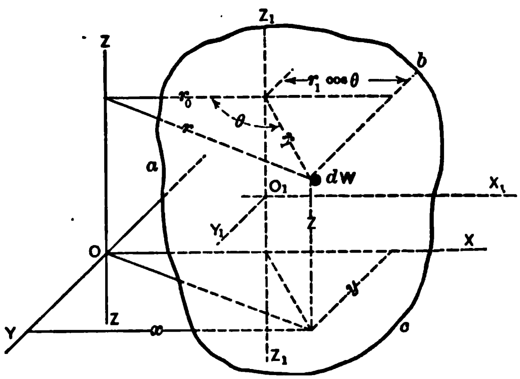

FIG. 146.

Let Z_1Z_1 be an axis parallel to ZZ passing through O_1, the center of gravity of the solid. Let dW be the weight of any elementary particle, the perpendicular distances of which from ZZ and Z_1Z_1 are r and r_1, respectively, and let r_0 be the perpendicular distance between the axes. Let θ equal the angle between r_1 and r_0.

The moment of inertia of the solid about the axis ZZ will be equal to

$$I = \int r^2\, dW,$$

and that about Z_1Z_1 will equal

$$I_0 = \int r_1^2\, dW.$$

From the triangle in the figure we have

$$r^2 = r_1^2 + r_0^2 - 2\, r_1 r_0 \cos\theta.$$

Hence

$$I = \int (r_1^2 + r_0^2 - 2\, r_1 r_0 \cos\theta)\, dW$$

$$= \int r_1^2\, dW + r_0^2 \int dW - 2\, r_0 \int r_1 \cos\theta\, dW. \quad . \quad . \quad (1)$$

Through the center of gravity O_1 assume an axis O_1X_1, in the plane of ZZ and Z_1Z_1. Then, if x_1 is the coördinate of the particle dW with respect to the X_1 plane through O_1, it is evident that

$$x_1 = -\, r_1 \cos\theta,$$

and from equation (1) we have

$$I = I_0 + r_0^2 W + 2\, r_0 \int x_1\, dW \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But, since Z_1Z_1 passes through the center of gravity, the quantity $\int x_1\, dW$, the moment of the weight with respect to the X_1 plane, will equal zero (Art. 89).

Therefore

$$I = I_0 + r_0^2 W. \quad . \quad . \quad . \quad . \quad (3) \text{ Q.E.D.}$$

It is evident that equation (2) will give the relation between the moments of inertia of a solid with respect to any two parallel axes. Since for the solid, $I = W\rho^2$ (Art. 131), if we let ρ = the radius of gyration about the axis OZ and ρ_0 = that about O_1Z_1 we shall have by substituting in equation (3)

$$W\rho^2 = W\rho_0^2 + Wr_0^2,$$

and hence

$$\rho^2 = \rho_0^2 + r_0^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the solid is in the form of a thin flat plate, perpendicular to ZZ, it is evident that this proposition gives the relation between the polar moments of inertia of the area of the plate about parallel axes. For, by dividing each term by the product of the weight

per unit volume and the thickness of the plate (Art. 132), equations (2) and (3) reduce to

$$I_z = I_{z_0} + r_0^2 A + 2 r_0 \int x_1 \, dA, \quad . \quad . \quad . \quad . \quad (5)$$

and

$$I_z = I_{z_0} + r_0^2 A, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where I_z and I_{z_0} are the polar moments of inertia of the area of the plate about the axes ZZ and Z_1Z_1, respectively, and A equals its area.

134. Problems. — Deduction of Formulas for Moments of Inertia of Homogeneous Solids. — In the solution of the following problems the solid in each case will be considered as made up of elementary slices. The moments of inertia of these slices about different axes through their centers of gravity may be determined from the formulas for moments of inertia of areas (Art. 113) by the method given in Art. 132. When necessary, the moment of inertia of a slice about an axis, parallel to the axis through its center of gravity, may be found by use of the formula

$$I = I_0 + r_0^2 W \text{ (Art. 133).}$$

In this manner the expressions for the moments of inertia of the elementary slices of a solid about different axes can be derived, the integration of which will give the formulas for the moments of inertia of the entire solid. References to the above-mentioned articles will be omitted.

Problem 1. — Thin Straight Rod of Uniform Section.

Deduce the formula for the moment of inertia of a thin straight rod about an axis through one end.

Solution. — If the cross section is so small that we can, without appreciable error, consider the mass of the rod as though it were concentrated along its center line, we may determine the formulas as follows:

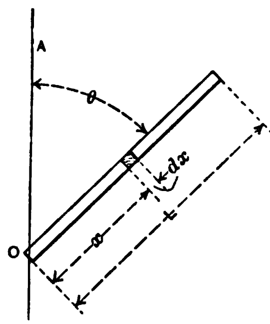

FIG. 147.

Let L = the length of the rod and θ = the angle it makes with the inertia axis OA (Fig. 147). Let w = the weight per unit length of the rod and W = the entire weight. Then $W = wL$.

If dx equals the length of an elementary particle, whose distance from O equals x, we have

$$dW = w\,dx,$$

and

$$r^2\,dW = x^2 \sin^2 \theta w\,dx.$$

Hence the moment of inertia about OA will be equal to

$$I = \int r^2 dW = w \sin^2\theta \int_0^L x^2 dx$$
$$= \frac{wL^3 \sin^2\theta}{3} = \frac{WL^2}{3} \sin^2\theta. \quad \ldots \ldots \quad (1)$$

When the rod is perpendicular to OA, $\theta = 90$ degrees,

and $$I = \frac{wL^3}{3} = \frac{WL^2}{3}. \quad \ldots \ldots \quad (2)$$

Substituting in equation (6), Art. 133, we obtain $I_0 = \frac{WL^2}{12}$ for the moment of inertia of the rod about an axis through its center of gravity and perpendicular to its length.

Problem 2. — Rectangular Prism.

Deduce the formula for the moment of inertia of the rectangular prism (Fig. 148) about the axis OZ, through the center of gravity perpendicular to the face.

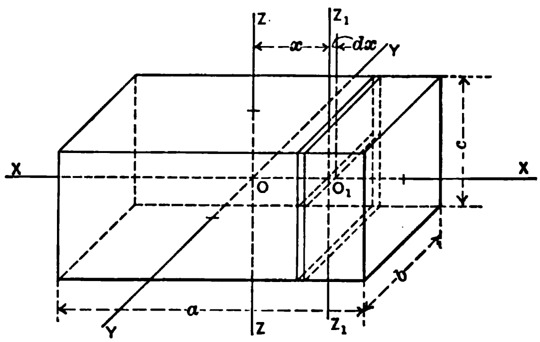

FIG. 148.

Solution. — Three solutions, illustrating three different methods of determining moments of inertia of solids, will be given.

Let the dimensions of the prism be a, b, c, and let w = its weight per unit volume and W = its entire weight. Then $W = wabc$.

Refer the prism to the three coördinate axes OX, OY, OZ, through its center of gravity, perpendicular to its faces.

(a) Consider the prism to be made up of elementary slices, of thickness dz, perpendicular to OZ. The moment of inertia of one of these slices about the axis OZ will be equal to

$$dI_z = w\frac{ab}{12}(a^2 + b^2)\, dz,$$

and the moment of inertia of the prism about OZ will be

$$I_z = \frac{wab}{12}(a^2 + b^2)\int_0^c dz = \frac{wabc}{12}(a^2 + b^2) = \frac{W}{12}(a^2 + b^2).$$

(b) Consider the prism to be made up of elementary slices, of thickness dx, perpendicular to OX (Fig. 148). The moment of inertia of one of these

slices about an axis O_1Z_1, through its center of gravity parallel to OZ, will be equal to $\frac{wcb^3}{12}\,dx$, and its moment of inertia about the axis OZ will be equal to

$$\frac{wcb^3}{12}\,dx + wbcx^2\,dx.$$

Then $$I_z = \frac{wcb^3}{12}\int_{-\frac{a}{2}}^{\frac{a}{2}} dx + wbc\int_{-\frac{a}{2}}^{\frac{a}{2}} x^2\,dx = \frac{wabc}{12}(a^2+b^2) = \frac{W}{12}(a^2+b^2).$$

(c) Consider the area to be divided into strips parallel to the X plane and determine the value of $\int x^2\,dW$ (Art. 129); in a similar manner determine the value of $\int y^2\,dW$; thus,

$$\int x^2\,dW = wbc\int_{-\frac{a}{2}}^{\frac{a}{2}} x^2\,dx = \frac{wbca^3}{12} \quad\text{and}\quad \int y^2\,dW = wac\int_{-\frac{b}{2}}^{\frac{b}{2}} y^2\,dy = \frac{wacb^3}{12}.$$

Having found the moments of inertia with respect to the X and Y planes we can determine I_z by finding their sum as follows:

$$I_z = \int x^2\,dW + \int y^2\,dW = \frac{wabc}{12}(a^2+b^2) = \frac{W}{12}(a^2+b^2).$$

Problem 3.—Right Circular Cylinder.

Deduce the formula for the moment of inertia of the right circular cylinder (Fig. 149): (a) about its axis XX; (b) about an axis YY, perpendicular to

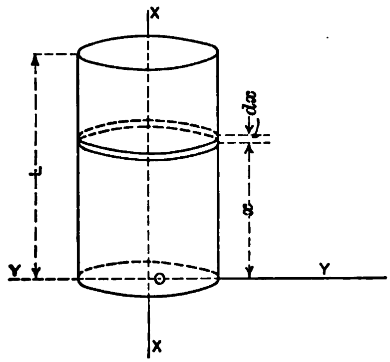

FIG. 149.

XX at the end of the cylinder; (c) about an axis perpendicular to XX, through the center of gravity.

Solution.—Let r = the radius, L = the length, w = the weight per unit volume and W = the entire weight. Then $W = w\pi r^2 L$.

(a) Moment of inertia about the axis XX.

The moment of inertia of a slice of thickness dx, perpendicular to XX, will be equal to

$$w\frac{\pi r^4}{2}\,dx.$$

Hence the moment of inertia of the cylinder will be equal to

$$I_x = \frac{w\pi r^4}{2}\int_0^L dx = \frac{w\pi r^4 L}{2} = \frac{Wr^2}{2}. \quad \ldots \ldots \quad (1)$$

(b) Moment of inertia about the axis YY.

The moment of inertia of the elementary slice about a diameter parallel to YY will be equal to $\frac{w\pi r^4}{4}dx$,

and its moment of inertia about YY,

$$\frac{w\pi r^4}{4}dx + w\pi r^2 x^2 dx.$$

Hence

$$\begin{aligned} I_y &= \frac{w\pi r^4}{4}\int_0^L dx + w\pi r^2 \int_0^L x^2 dx \\ &= \frac{w\pi r^4 L}{4} + \frac{w\pi r^2 L^3}{3} = w\pi r^2 L\left(\frac{r^2}{4} + \frac{L^2}{3}\right) \\ &= W\left(\frac{r^2}{4} + \frac{L^2}{3}\right). \quad \ldots \ldots \quad (2) \end{aligned}$$

It is evident if r is so small compared with L that the term $\frac{r^2}{4}$ may be omitted from equation (2) without introducing a serious error, the formula will reduce to that for the straight rod (Prob. 1).

(c) Moment of inertia about an axis through the center of gravity of the cylinder, perpendicular to XX.

Substituting in the equation $\quad I = I_0 + r_0^2 W$

$$W\left(\frac{r^2}{4} + \frac{L^2}{3}\right) = I_0 + W\left(\frac{L}{2}\right)^2.$$

Hence

$$I_0 = \frac{W}{4}\left(r^2 + \frac{L^2}{3}\right). \quad \ldots \ldots \quad (3)$$

Problem 4. — Right Elliptic Cylinder.

(a) Prove that the moment of inertia of the right elliptic cylinder about its axis XX will equal

$$I_x = \frac{w\pi abL}{4}(a^2 + b^2) = \frac{W}{4}(a^2 + b^2), \quad \ldots \ldots \quad (1)$$

where a and b are the semi-major and semi-minor axes of the ellipse; L, the length of the cylinder; w, its weight per unit volume; and W, its entire weight.

(b) Prove that the moment of inertia of the right elliptic cylinder about an axis through its center of gravity, coinciding with the minor axis $2b$ of its cross section, will be equal to

$$I_0 = \frac{w\pi abL}{4}\left(a^2 + \frac{L^2}{3}\right) = \frac{W}{4}\left(a^2 + \frac{L^2}{3}\right). \quad \ldots \ldots \quad (2)$$

Problem 5. — Thin Hollow Circular Cylinder.

(a) Prove that the moment of inertia of the thin hollow cylinder about its axis XX is equal to

$$I_x = Wr^2, \quad \ldots \ldots \quad (1)$$

where W equals the total weight of the cylinder and r equals its radius.

(b) Prove that the moment of inertia of the thin hollow cylinder about an axis through its center of gravity, perpendicular to the axis of the cylinder, will be equal to

$$I_0 = \frac{W}{2}\left(r^2 + \frac{L^2}{6}\right), \quad \ldots \ldots \ldots \quad (2)$$

where L equals the length of the cylinder.

Problem 6. — Ellipsoid.

Deduce the formula for the moment of inertia of an ellipsoid about its major axis $2a$ (Fig. 150): given the equation of the ellipsoid, referred to the axes, OX, OY and OZ, coinciding respectively with its major, minor and intermediate axes, $2a$, $2b$ and $2c$,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Solution. — Let w = the weight per unit of volume and W = the entire weight of the ellipsoid.

Then $W = \frac{4}{3} w\pi abc$ (Prob. 3, Art. 104).

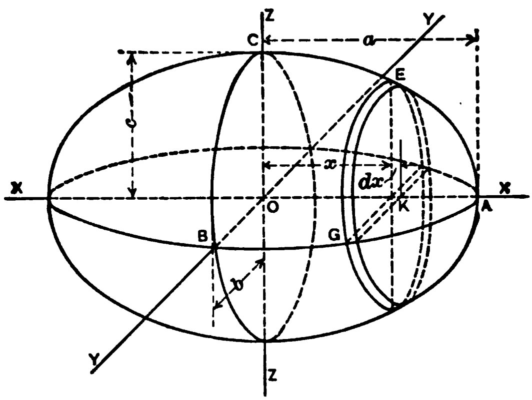

FIG. 150.

Consider the ellipsoid to be divided into thin slices, of the thickness dx, perpendicular to OX. The semi-diameters EK and GK of one of these slices, whose distance from O is equal to x, will be

$$EK = \frac{c}{a}\sqrt{a^2 - x^2} \quad \text{and} \quad GK = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Its moment of inertia with respect to OX will be equal to

$$\frac{w\pi bc}{4a^2}(a^2 - x^2)\left[\frac{b^2}{a^2}(a^2 - x^2) + \frac{c^2}{a^2}(a^2 - x^2)\right]dx$$

$$= \frac{w\pi bc\,(b^2 + c^2)}{4a^4}(a^2 - x^2)^2.$$

Hence the moment of inertia of the ellipsoid about OX will be

$$I_x = \frac{w\pi bc\,(b^2+c^2)}{4\,a^4}\int_{-a}^{a}(a^4 - 2\,a^2x^2 + x^4)dx$$
$$= \frac{4}{15}\,w\pi abc\,(b^2+c^2) = \frac{W}{5}\,(b^2+c^2).$$

Problem 7. — Sphere.

Deduce the formula for the moment of inertia of the sphere about its diameter.

Solution. — By substituting $a = b = c = r$ in the formula for the ellipsoid we obtain for the moment of inertia of the sphere, about its diameter,

$$I = \frac{8}{15}\,w\pi r^5 = \frac{2}{5}\,Wr^2.$$

Problem 8.

Deduce the formula for the moment of inertia of the sphere directly by the method of integration.

Problem 9. — Thin Hollow Sphere.

Prove that the moment of inertia of the thin hollow sphere about its diameter is equal to

$$I = \frac{2}{3}\,Wr^2,$$

where W = the total weight of the sphere and r = its radius.

Find its moment of inertia with respect to its center and note that the moments of inertia with respect to the three coördinate planes through O must be equal (Art. 129).

Problem 10. — Right Circular Cone.

(a) Prove that the moment of inertia of a right circular cone about its axis is equal to

$$I = \frac{3}{10}\,Wr^2, \qquad (1)$$

where W = its total weight and r = the radius of the base.

(b) Prove that the moment of inertia about an axis through the apex, perpendicular to the axis of the cone, is equal to

$$I = \frac{3}{5}\,W\left(\frac{r^2}{4} + h^2\right), \qquad (2)$$

where h = the altitude of the cone.

135. Summary of Formulas for Moments of Inertia of Solids. — For purposes of reference the following list of the formulas deduced in Art. 134 for the moments of inertia, in terms of the total weights, W, is given.

$\frac{WL^2}{3}\sin^2\theta$. (Thin straight rod, axis through one end making angle θ with rod.)

$\frac{WL^2}{3}$. (Thin straight rod, axis perpendicular to rod at one end.)

$\frac{WL^2}{12}$. (Thin straight rod, axis perpendicular to rod at center.)

$\frac{W}{12}(a^2 + b^2)$. (Rectangular prism, axis through center of gravity perpendicular to face.)

$\frac{Wr^2}{2}$. (Circular cylinder, about its axis.)

$W\left(\frac{r^2}{4} + \frac{L^2}{3}\right)$. (Circular cylinder, axis perpendicular to axis of cylinder at one end.)

$\frac{W}{4}\left(r^2 + \frac{L^2}{3}\right)$. (Circular cylinder, axis through center of gravity, perpendicular to axis of cylinder.)

$\frac{W}{4}(a^2 + b^2)$. (Elliptical cylinder, about axis of cylinder.)

$\frac{W}{4}\left(a^2 + \frac{L^2}{3}\right)$. (Elliptical cylinder, axis through center of gravity coinciding with minor axis, $2b$, of cross section.)

Wr^2. (Thin hollow circular cylinder, about axis of cylinder.)

$\frac{W}{2}\left(r^2 + \frac{L^2}{6}\right)$. (Thin hollow circular cylinder, axis through center of gravity perpendicular to axis of cylinder.)

$\frac{W}{5}(b^2 + c^2)$. (Ellipsoid, about major axis, $2a$.)

$\frac{2}{5}Wr^2$. (Sphere, about diameter.)

$\frac{2}{3}Wr^2$. (Thin hollow sphere, about diameter.)

$\frac{3}{10}Wr^2$. (Right circular cone, about its axis.)

$\frac{3}{5}W\left(\frac{r^2}{4} + h^2\right)$. (Right circular cone, about axis through its apex, perpendicular to axis of cone.)

136. Method of Determining Moments of Inertia of Solids by Dividing into Finite Parts. — As in the case of plane areas (Art. 114) the moments of inertia of solids may in some cases be conveniently determined by dividing into parts and adding together the moments of inertia of the parts.

In other cases the moments of inertia may be determined by finding the difference of the moments of inertia of two solids.

137. Problems. — Moments of Inertia of Solids.

Problem 1.

Find the moment of inertia of a homogeneous rod, 5′ long and of rectangular cross section (2″ × 3″), about its axis XX (Fig. 151).

Find its moment of inertia about an axis ZZ, perpendicular to XX at the end of the rod; also about an axis parallel to ZZ, through the center of gravity. Weight of rod = 180 lbs.

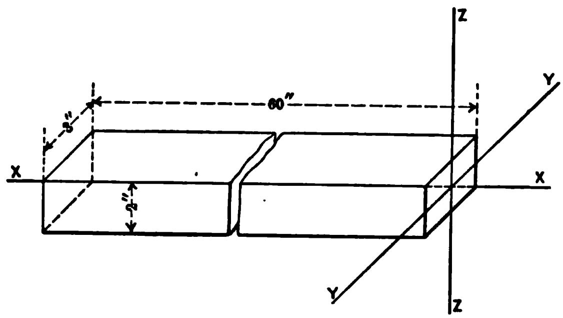

Fig. 151.

Problem 2.

(a) Find the moment of inertia of the solid (Fig. 118) about the axis YY through the center of the sphere; also about the axis XX.

(b) Find the moment of inertia about the axis perpendicular to XX through the center of gravity. Weight of material = ¼ lb. per cu. in.

Problem 3.

Find the moment of inertia of the flywheel (Fig. 152) about its axis. Weight of material = 450 lbs. per cu. ft.

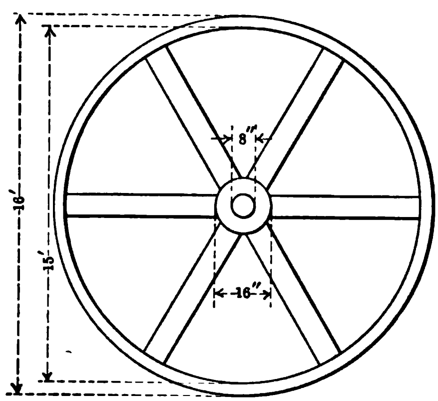

Fig. 152.

Width of hub and rim = 20″.

Area of cross section of each arm = 40 (sq. ins.).

Give results in lbs. (ft.)^2 and also in lbs. (ins.)^2.

In this problem the moments of inertia of the rim and hub may be found by using the formula for the cylinder (Art. 135). The moment of inertia of the arms may be found with a small approximation by treating each arm as a rod of uniform cross section.

Problem 4.

Find the moment of inertia of the solid (Fig. 153) about its axis XX; also about the axis YY, perpendicular to XX through the end of the solid. Weight of material = 0.3 lb. per cu. in.

This problem should be solved by integration.

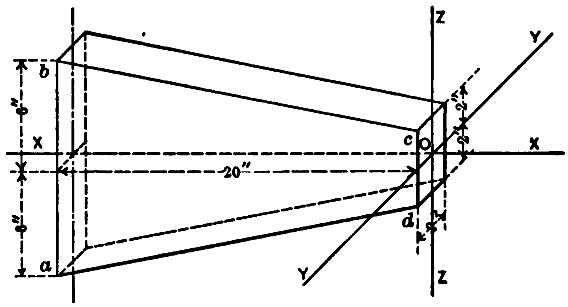

Fig. 153.

Problem 5.

Find the moment of inertia of a wedge 6″ wide, 12″ long and 4″ thick at the large end, about an axis through its cutting edge. Weight of material = 0.3 lb. per cu. in.

Problem 6.

Find the moment of inertia of a solid of revolution, of the cross section shown in Fig. 114f, about its axis YY; also about the axis XX perpendicular to YY. Weight of material = 0.3 lb. per cu. in.

Problem 7.

Find the moment of inertia of a cube about a diagonal. Length of side = c. Weight of material per unit volume = w. (See Art. 139.)

138. Product of Inertia of a Solid. — The product of inertia of a solid is the sum of the products obtained by multiplying the weight of each of the elementary particles, into which it may be conceived to be divided, by the product of its distances from two of the three coördinate planes through a given point.

Referring to Fig. 144 the products of inertia of the solid with respect to the X and Y planes will be

$$K_{xy} = \int xy\, dW;$$

with respect to the Y and Z planes will be

$$K_{yz} = \int yz\, dW;$$

and with respect to the X and Z planes will be

$$K_{xz} = \int xz\, dW.$$

The products of inertia may also be expressed in terms of the mass as follows:

$$K_{xy_m} = \int xy\, dM, \quad K_{yz_m} = \int yz\, dM, \quad K_{xz_m} = \int xz\, dM.$$

It is evident that $K_{xy_m} = \dfrac{K_{xy}}{g}$, etc.

By a method, similar to that used in the case of the moment of inertia (Art. 132), it may be shown that the product of inertia of a thin flat plate, of uniform thickness and material, with respect to a pair of rectangular coördinate axes, OX and OY, in the plane of the plate will be equal to

$$K_{xy} = \int xy\, dw = wt \int xy\, dA,$$

that is: the product of inertia of the plate is equal to the product of inertia of its area multiplied by the product of the weight per unit of volume and the thickness. Hence the product of inertia of any solid may be found by dividing it into slices of infinitesimal thickness and obtaining the sum of the products of inertia of the slices by integration.

By a method similar to that employed in the case of the plane area (Art. 123), we may determine the product of inertia of any solid by finding the algebraic sum of the products of inertia of the finite parts into which it may be conveniently divided.

The units in which the products of inertia of a solid body are expressed will evidently be the same as the units of moments of inertia.

139. Product of Inertia of a Solid when one of the Coördinate Planes is a Plane of Symmetry. — *Proposition:* — *The product of inertia of a homogeneous solid with respect to two planes at right angles to each other, one of which is a plane of symmetry, is equal to zero.*

Let *abc* be any solid, which is symmetrical with respect to the X plane, the axis OZ being perpendicular to the plane of the paper (Fig. 154).

Any section of the solid, parallel to the Z plane, will be symmetrical with respect to its line of intersection with the X plane.

Let *def* represent such a section. The product of inertia of the area *def* with respect to its lines of intersection with the X and Y planes will be equal to

$$\int xy\,dA = 0. \quad \text{(Art. 122.)}$$

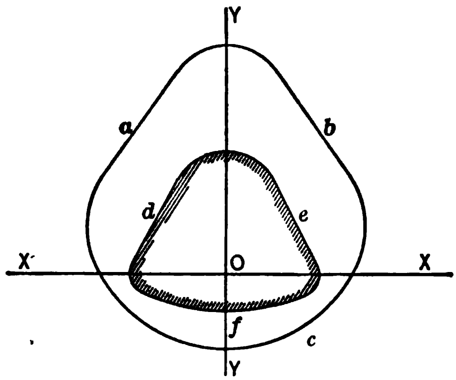

FIG. 154.

Hence the product of inertia of a thin slice *def* of the thickness dz, with respect to the X and Y planes, will be equal to

$$w\,dz\int xy\,dA = 0.$$

Since the product of inertia of each slice is equal to zero, the product of inertia of the whole body with respect to the X and Y planes will be equal to zero, or

$$K_{xy} = 0. \qquad \text{Q.E.D.}$$

By similar reasoning we may show that the product of inertia of the solid with respect to the X and Z planes will also be equal to zero, or

$$K_{xz} = 0.$$

Hence, if one of three planes at right angles to each other, passing through a given point, is a plane of symmetry for a given body, the product of inertia of the body with respect to the plane of symmetry and each of the other coördinate planes will be equal to zero.

It is evident if two of the coördinate planes are planes of symmetry the three products of inertia are equal to zero.

140. Principal Moments of Inertia of Solids. — By methods similar to those used in the case of the plane area, the following propositions may be deduced:

(a) The moment of inertia of a solid about any axis passing through a given point can be obtained in terms of the moments of inertia and products of inertia for three rectangular axes passing through that point, when the angles which the given axis makes with the three rectangular axes are known.

Thus, if I_x, I_y, I_z, K_{xy}, K_{xz} and K_{yz} are the moments of inertia and products of inertia with respect to three rectangular axes OX, OY and OZ, it can be proved that the moment of inertia I about an axis through O making angles α, β and γ with OX, OY and OZ, respectively, will be equal to

$$\begin{aligned} I = {} & I_x \cos^2 \alpha + I_y \cos^2 \beta + I_z \cos^2 \gamma \\ & -2\, K_{xy} \cos \alpha \cos \beta - 2\, K_{xz} \cos \alpha \cos \gamma \\ & -2\, K_{yz} \cos \beta \cos \gamma \quad \dots \dots \dots \quad (1) \end{aligned}$$

(b) There are three rectangular axes passing through any given point, called the principal axes, about one of which the moment of inertia of the solid is a maximum and about another a minimum, the moment of inertia about the third axis being intermediate in value between the other two, or equal to either one of them.

The planes perpendicular to these axes are called principal planes.

(c) The products of inertia of the solid with respect to the principal planes are equal to zero.

Hence the moment of inertia of the solid about any axis, making

angles α, β and γ, respectively, with the principal axes OX, OY and OZ through any point, will be given by the expression

$$I = I_x \cos^2 \alpha + I_y \cos^2 \beta + I_z \cos^2 \gamma \quad . \quad . \quad . \quad (2)$$

The axes at the intersections of three planes at right angles to each other, two of which are planes of symmetry, will be principal axes (Art. 139).

(d) The relations between the moments of inertia of a solid about different axes passing through any point may be expressed geometrically by means of an ellipsoid of inertia, or momental ellipsoid, corresponding to the ellipse of inertia for the plane area (Art. 127).

(e) When the moments of inertia of a solid about the three principal axes, passing through a given point, are equal, the moments of inertia about all axes, passing through that point, are equal.

(f) When the moments of inertia of a solid about two of the principal axes are equal, the moments of inertia about all axes, in the plane of those two and passing through the origin, are equal.

CHAPTER V.

KINETICS.

§1. Kinematics: Kinetics of the Particle.

141. Effect of Forces in Producing or Modifying Motion.— In determining the effect of a force, or system of forces, in producing or modifying motion, we may consider the body, on which the forces act, as being made up of a system of particles, so small that each one of them may be treated as though its mass were concentrated at a point. Any one of these particles will be subjected to the action of a system of forces, exerted upon it by the other particles in the body and also by outside bodies.

In accordance with the Laws of Motion (Art. 11) we may state the following:

(*a*) If the particle is at rest, or moving uniformly in a straight line, the resultant of the forces acting upon it will be equal to zero.

(*b*) If it moves with an accelerated motion in a straight line, the resultant will act in the line of motion and be equal to the product of the mass of the particle and its acceleration.

(*c*) If it moves in a curve, the resultant will act so as to produce the necessary deviation of the path from a straight line, as well as the acceleration, if any, of the particle in its path.

Hence, if we know the acceleration of a particle, we can determine its accelerating force, which will be equal to the resultant of all the forces exerted upon it. Since the "action" and "reaction" between any two particles in a body must balance, the resultant of the forces acting on all the particles of a body must be equal to the resultant of the external forces (Art. 12) acting on the body.

Therefore we may determine the resultant of the external forces necessary to impart a given acceleration to any body by finding the resultant of the accelerating forces necessary to impart the required motion to each of the particles, of which the body is composed.

This may be regarded as a somewhat modified statement of D'Alembert's "Principle," which may be given in the following form: If forces, equal and opposite to the changes in their mo-

menta, per unit of time, are applied to the elementary particles into which any mass may be conceived to be divided, these, together with the external forces acting on the body, will form a balanced system.

Before proceeding with the discussion of the effects of forces on the motion of a rigid body, we will consider certain fundamental principles governing the motion of the particle, when the path in which it moves lies in a single plane; in other words, a particle having *plane motion* only.

142. Linear Velocity. — The general expression for the *linear velocity* of a particle is

$$v = \frac{ds}{dt} \text{(Art. 7)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the particle moves in a straight line its motion is *rectilinear*, and its velocity at every instant is the same in direction. If the path in which the particle moves is a curve its motion is *curvilinear*, and its velocity changes in direction at every instant.

A distinction should be made between *velocity* and *speed*, the velocity at any instant being a vector quantity indicating the rate and direction of the motion, while the speed is a scalar quantity indicating the rate of motion of the particle in its path without regard to the direction. In other words, the speed is the *magnitude* of the velocity. Since the velocity of a particle is a vector quantity, it follows that it may be resolved into components by the methods of resolving vectors (Art. 146): and, conversely, when the components of the velocity are known, the resultant velocity of the particle may be obtained by finding their vector sum.

It follows from the definition that the resultant velocity of a particle at any instant will be in the direction of its motion at that instant.

When the speed is constant, the motion of the particle is said to be *uniform* and the space s, passed over in the time t, will be equal to

$$s = vt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

When the speed is not constant, the motion is said to be *variable* and the formula for the space passed over in the time t becomes

$$s = \int_0^t v\,dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In the British system of units, linear velocity is usually expressed in *feet per second*, which may be abbreviated, ft. per sec.

If a diagram is made by plotting the displacement at the end of any given time as an ordinate and the corresponding time as an abscissa, the resulting graph will show the relation between the space and time at any instant, and the slope of the graph at any point will represent the velocity at that point, the actual velocity being equal to the slope multiplied by a constant depending on the scale of the plot.

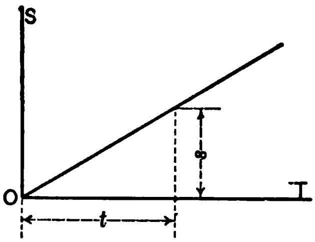

Fig. 155.

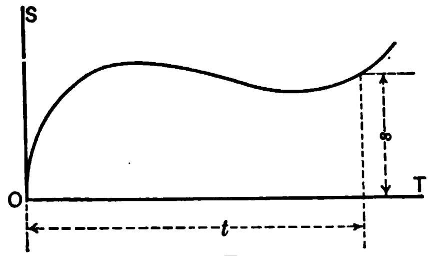

Fig. 155a.

A diagram for uniform motion is shown in Fig. 155, and for variable motion in Fig. 155a.

The foregoing will evidently apply to the motion of a particle with respect to a moving point as well as to its motion with respect to a so-called "fixed point" on the Earth's surface (Art. 6).

143. Linear Acceleration. — *Rectilinear Motion.* — When the motion of a particle is rectilinear, the general expression for its acceleration is

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} \text{ (Art. 8)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When the acceleration is constant the motion of the particle is said to be *uniformly varying*. By integrating equation (1) and letting v_0 = the initial velocity, when $t = 0$, we obtain

$$v = \frac{ds}{dt} = v_0 + at. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By integrating equation (2) we obtain

$$s = v_0t + \frac{1}{2}at^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where s = the space passed over in the time t.

✓ Eliminating t between equations (2) and (3) we have

$$v = \sqrt{v_0^2 + 2\,as}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

When the acceleration is not constant, the motion of the particle may be said to be *non-uniformly varying*, in which case we obtain by integrating equation (1)

$$v = \frac{ds}{dt} = v_0 + \int_0^t a\,dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $a = \phi\,(t)$, and

$$s = \int_0^t v\,dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Also from equation (5)

$$t = \int_{t=0}^{t=t} \frac{ds}{v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In the British system of units, linear acceleration is usually expressed in *feet per second per second*, which may be abbreviated, ft. per sec.^{2}.

If a diagram is made by plotting the velocity at any time as an ordinate and the time as an abscissa, the resulting graph will show the relation between the velocity and time at any instant, and the slope of the graph at any point will represent the acceleration at that point, the actual acceleration being equal to the slope multiplied by a constant depending on the scale of the plot.

A diagram for uniformly varying motion is shown in Fig. 156 and for non-uniformly varying motion in Fig. 156a.

The space-time curve (Art. 142) may be determined from the velocity-time curve in the following manner. When the distances are measured from the starting point, that is, where $t = 0$, the ordinate to the former curve at the end of any time t will be equal to the area under the latter curve between the ordinates $t = 0$ and $t = t$, plotted to a convenient scale. Hence, if the area under the velocity-time curve is divided into narrow strips and the area of each strip calculated by multiplying its mean ordinate by its width, the successive ordinates of the space-time curve will be equal to the partial sums obtained by adding the areas of the strips under the velocity-time curve between the ordinate when $t = 0$ and the ordinates at the successive values of t.

If distances are measured from some other point than the one indicated above, the ordinates of the space-time diagram will be

equal to the ordinates obtained in the preceding manner plus, or minus, a constant, the constant being equal to the space passed over when $t = 0$; that is, when the body has reached the point from which the time is measured.

If a diagram is made in which the *acceleration* at any instant is plotted as an ordinate and the corresponding time as an abscissa,

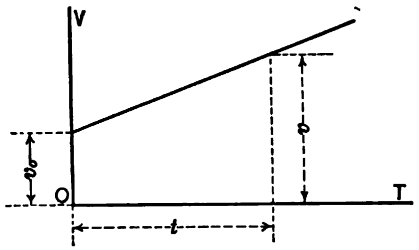

FIG. 156.

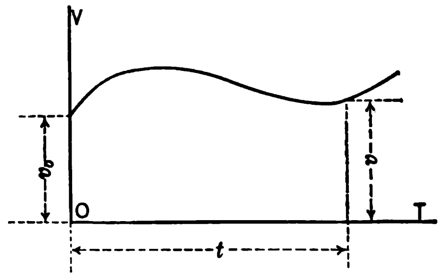

FIG. 156a.

the velocity-time plot may be obtained from the acceleration-time plot in the same manner as the space-time plot is obtained from the velocity-time plot.

Since the acceleration of a particle is a quantity having both magnitude and direction, it may be represented by a vector, and accelerations may be resolved into components and combined in the same manner as velocities (Art. 146).

Curvilinear Motion.—When a particle moves in a curved path with a constant or varying speed, the direction of its motion is constantly changing. Since acceleration is the rate of change of velocity, the motion of such a particle is always accelerated, whether its speed is uniform or varying.

Let v, v_1, v_2, v_3 be the linear velocities at the points b, c, d, e (Fig. 156b) of a particle moving in a curved path. From some point O lay off the vectors OV, OV_1, OV_2, OV_3, representing the velocities v, v_1, v_2, v_3, respectively, and draw the curve $VV_1V_2V_3$. Then, if we let bc represent an element Δs of the path described in the time Δt, the velocity v_1 will be the resultant of the velocity v and the velocity VV_1, acquired during the time Δt; and the acceleration of the particle at the point b will be equal to

$$a = \text{limit of } \frac{VV_1}{\Delta t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The curve $VV_1V_2V_3$ is called the *hodograph* of the motion and it is evident that if the curve is described by a point V, moving so as to be always at the end of the vector representing the velocity of the particle in its path, the velocity of V will always be equal to the acceleration of the particle. If we represent the accelera-

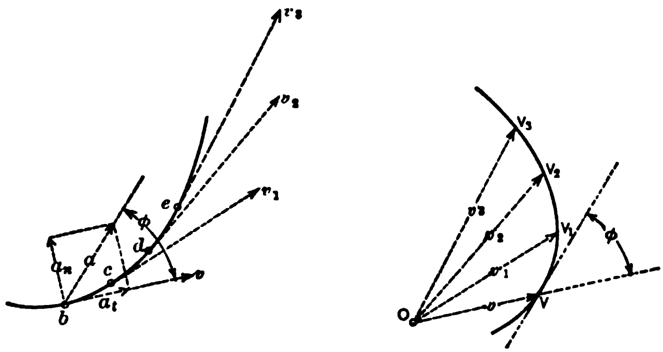

FIG. 156b.

tion at b by the vector a and let ϕ = the angle between a and the tangent to the path, we may resolve the acceleration into a component

$$a_n = a \sin \phi,$$

in the direction of the normal to the path, and a component

$$a_t = a \cos \phi,$$

in the direction of the tangent.

If we let $v_1 - v = \Delta v$, r = the radius of curvature at b, $\Delta\theta$ = the angle VOV_1 between the tangents at b and c, and resolve the increment VV_1 into components parallel to a_t and a_n, we shall have for the component parallel to a_t,

$$VV_1 \cos \phi = \Delta v \text{ (very nearly)}, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and for the component parallel to a_n,

$$VV_1 \sin \phi = v\,\Delta\theta = v\frac{\Delta s}{r} \text{ (very nearly)}. \quad . \quad . \quad . \quad (10)$$

Dividing equations (9) and (10) by Δt and passing to the limit we have the *tangential* component of the acceleration,

$$a \cos \phi = \frac{dv}{dt} = \frac{d^2s}{dt^2} = a_t, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and

$$a \sin \phi = \frac{v}{r}\frac{ds}{dt} = \frac{v^2}{r} = a_n. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

the *normal* component of the acceleration. Hence the expression for the resultant acceleration of a particle moving in any curve will be

$$a = \sqrt{a^2_t + a^2_n} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \frac{v^4}{r^2}}. \quad . \quad . \quad . \quad . \quad (13)$$

As in the case of velocities, the foregoing will evidently apply to the motion of a particle relatively to a moving point as well as to its motion in relation to a "fixed point."

144. Angular Velocity. — If the motion of a particle be referred to some point O, not in its path, then a straight line, joining O and the particle, will have an angular displacement θ during a given displacement of the particle. The amount of the angular motion θ will in general be different for different points of reference.

The rate at which the angular displacement occurs at any instant is the *angular velocity* of the particle with respect to the point O at that instant. Hence the magnitude of the angular velocity will be equal to

$$\omega = \frac{d\theta}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $d\theta$ = the angular displacement during the time dt.

A distinction should be made between the quantity ω, representing the magnitude, or scalar part, of the angular velocity, which we may call *angular speed*, and the angular velocity, which may be represented by a vector, laid off along an axis perpendicular to the plane of the path of the particle, the direction of the motion being indicated in a manner similar to that in the case of the couple (Art. 64).

When the angular velocity is constant, the angular motion of the particle, during the time t, will be equal to

$$\theta = \omega t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

When the angular velocity is variable

$$\theta = \int_0^t \omega\, dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $\omega = \phi(t)$.

If the particle moves in a *plane* curve and v equals its linear velocity at any point and r = the radius of curvature of the path at that point, its *angular velocity* about the center of curvature will be equal to

$$\omega = \frac{d\theta}{dt} = \frac{1}{r}\frac{ds}{dt} = \frac{v}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Transposing we obtain the expression for its linear velocity,

$$v = \frac{ds}{dt} = r\omega = r\frac{d\theta}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If the motion is referred to a point O in the plane of the path, which is not the center of curvature, the resultant velocity v may be resolved into a component,

$$v_1 = \frac{dr}{dt},$$

along the radius vector r, between the point and the particle; and a component

$$v_2 = r\frac{d\theta}{dt},$$

perpendicular to r. In this case the expression for the resultant linear velocity of the particle may be written

$$v = \frac{ds}{dt} = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Where the quantity $\frac{d\theta}{dt}$ may be called the angular velocity of the particle in respect to O.

It is evident from the above that, when the second is the unit of time, angular velocity will be expressed in *radians per second*, which may be abbreviated, rads. per sec.

A diagram may be made showing the relation between the angular motion θ and the time t, in the same manner as the diagrams for linear motion (Art. 142).

145. Angular Acceleration. — The rate of change of the angular velocity of a particle is its angular acceleration. Denoting this quantity by α, we have

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When the *angular acceleration is constant,* we obtain by integrating equation (1), and letting ω_0 = the initial angular velocity when $t = 0$,

$$\omega = \frac{d\theta}{dt} = \omega_0 + \alpha t, \quad \ldots \ldots \quad (2)$$

and, by integrating equation (2),

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2, \quad \ldots \ldots \quad (3)$$

where θ = the angular motion during the time t.

Eliminating t between equations (2) and (3) we have

$$\omega = \sqrt{\omega_0^2 + 2\alpha\theta}. \quad \ldots \ldots \quad (4)$$

When the *angular acceleration is varying,* we obtain by integrating equation (1)

$$\omega = \omega_0 + \int_0^t \alpha\, dt, \quad \ldots \ldots \quad (5)$$

where $\alpha = \phi(t)$, and

$$\theta = \int_0^t \omega\, dt. \quad \ldots \ldots \quad (6)$$

Substituting in equation (1) the value of ω (equation 4, Art. 144) we obtain the expression for the angular acceleration of the particle with respect to the center of curvature of its path,

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{1}{r}\frac{ds}{dt}\right). \quad \ldots \ldots \quad (7)$$

When the *path is circular,* r = a constant and

$$\alpha = \frac{d\omega}{dt} = \frac{1}{r}\frac{d^2s}{dt^2} = \frac{a_t}{r}, \quad \ldots \ldots \quad (8)$$

that is, the angular acceleration is equal to the quotient obtained by dividing the tangential acceleration (Art. 143) by the radius of the circle, and conversely,

$$a_t = r\alpha = r\frac{d\omega}{dt}. \quad \ldots \ldots \quad (9)$$

It is evident from the preceding that the angular acceleration of a particle is a quantity which may be represented by a vector

in a manner similar to angular velocity, and that a diagram might be made showing the relation between the angular velocity and the time, in the same manner as in the case of linear motion (Art. 143).

It is evident from equation (1) that in any system of units in which the second is the unit of time the angular acceleration will be expressed in *radians per second per second,* which may be abbreviated, rads. per sec.^{2}.

146. Resolution and Composition of Velocities.—If the path of a particle be referred to any two rectangular axes, OX and OY, in the plane of its motion, the resultant velocity v at any point A (Fig. 157) may be resolved into components v_x and v_y parallel to OX and OY, respectively, (Art. 142).

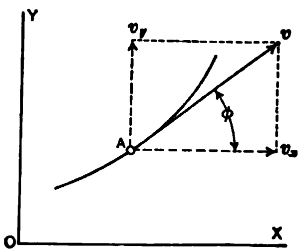

FIG. 157.

If ϕ = the angle between the vector v and the axis OX,

$$v_x = v \cos \phi,$$

and

$$v_y = v \sin \phi.$$

But $$v = \frac{ds}{dt}, \quad \cos \phi = \frac{dx}{ds}, \quad \sin \phi = \frac{dy}{ds},$$

and therefore

$$v_x = \frac{dx}{dt}, \quad \ldots \ldots \ldots \quad (1)$$

and

$$v_y = \frac{dy}{dt}. \quad \ldots \ldots \ldots \quad (2)$$

Hence if the coördinates x and y of any point A in the path are expressed in terms of t, the components of the velocity may be found by differentiation.

Conversely, if v_x and v_y are known, the resultant velocity will be equal to

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}. \quad . \quad . \quad . \quad (3)$$

If the particle has imparted to it any number of velocities simultaneously, we may resolve each velocity into components parallel to two axes OX and OY, as indicated above. The resultant velocity will then be equal to

$$v_r = \sqrt{(\Sigma v_x)^2 + (\Sigma v_y)^2},$$

where Σv_x and Σv_y are the vector sums of the components parallel to OX and OY respectively.

If OX and OY were taken parallel to the tangent and normal to the path at A, respectively, it is evident that Σv_y would be equal to zero.

The following proposition concerning the relative velocities of two particles is important.

If at any instant the velocity v_b of a particle B, relatively to a particle A, is known and at the same time the velocity v_a of A, relatively to an origin O; the resultant velocity of B, relatively to O, will be equal to the vector sum of the velocities v_b and v_a.

The following is the proof of the above when the particles move in the same plane. Assume the rectangular coördinate axes OX

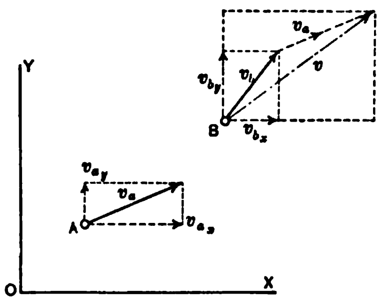

FIG. 157a.

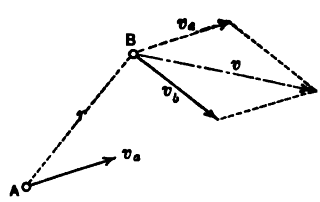

FIG. 157b.

and OY in the plane of motion (Fig. 157a) and resolve v_b into components v_{b_x} and v_{b_y} parallel to OX and OY. Then v_{b_x} and v_{b_y} would be the components of the velocity of B if the point A were "fixed." Since A is moving relatively to O with a velocity v_a

in the direction shown, the components of the velocity of B relatively to O in the directions of the axes OX and OY will be equal to

$$v_x = v_{a_x} + v_{b_x},$$

and

$$v_y = v_{a_y} + v_{b_y}.$$

where v_{a_x} and v_{a_y} are the components parallel to OX and OY of the velocity of A relatively to O.

Hence the resultant velocity of B relatively to O will be equal to

$$v = \sqrt{(v_{a_x} + v_{b_x})^2 + (v_{a_y} + v_{b_y})^2}, \quad . \quad . \quad . \quad . \quad (4)$$

and it is evident from the figure that v is the vector sum of v_a and v_b.

A special case arises when the distance between A and B is invariable; in which case the velocity of B relatively to A at any instant must be perpendicular in direction to the straight line AB joining the particles (Fig. 157b); that is, the path of B relatively to A must be a circle. If we let $AB = r$,

$$v_b = \omega r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where ω = the angular velocity of B with respect to A (Art. 144).

Conversely, if in this case the resultant velocities of B and A relatively to a fixed point O are known, the velocity of B relatively to A may be determined as indicated in Fig. 157b.

147. Resolution and Composition of Accelerations. — In Art. 143 it was shown that the resultant acceleration of a particle moving in a curved path may always be represented by a vector lying on the *concave* side of the curve and that this acceleration, expressed in terms of the normal and tangential components, will be equal to

$$a = \sqrt{a_n^2 + a_t^2} = \sqrt{\left(\frac{v^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2} = \sqrt{\frac{1}{r^2}\left(\frac{ds}{dt}\right)^4 + \left(\frac{d^2s}{dt^2}\right)^2}. \quad (1)$$

When the motion is rectilinear, equation (1) reduces to

$$a = \frac{d^2s}{dt^2} = a_t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The acceleration of the particle may be resolved into components parallel to the coördinate axes OX and OY in the same

manner as the velocity (Art. 146). Thus, if the resultant acceleration of the particle at A (Fig. 158) is represented by the vector a, it may be resolved into components a_x and a_y parallel to OX and OY respectively, and, if β = the angle which the vector a makes with OX,

$$a_x = a \cos \beta, \quad \text{and} \quad a_y = a \sin \beta.$$

But a_x will be equal to the rate of change of the component of the velocity of A in the direction of OX (Art. 146), and hence,

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In a similar manner a_y will be equal to the rate of change of the component of the velocity in the direction OY, and hence,

$$a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

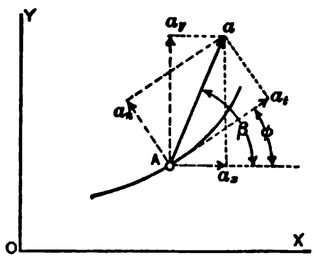

FIG. 158.

Conversely, the resultant acceleration expressed in terms of its components in any two directions at right angles will be equal to

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}. \quad . \quad . \quad . \quad (5)$$

Equating (1) and (5) we have for curvilinear motion

$$a = \sqrt{\frac{1}{r^2}\left(\frac{ds}{dt}\right)^4 + \left(\frac{d^2s}{dt^2}\right)^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}, \quad . \quad . \quad (6)$$

and equating (2) and (5) we have for rectilinear motion

$$a = \frac{d^2s}{dt^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}. \quad . \quad . \quad . \quad . \quad (7)$$

When the path is a circle

$$a_t = r\alpha \quad \text{and} \quad a_n = r\omega^2.$$

and hence for this case equation (1) may be written

$$a = \sqrt{a_t^2 + a_n^2} = r\sqrt{\alpha^2 + \omega^4}. \quad . \quad . \quad . \quad . \quad (8)$$

The following proposition concerning the relative accelerations of two particles is important.

If at any instant the acceleration a_b of a particle B, relatively to a particle A, is known and at the same time the acceleration a_a of A, relatively to an origin O, then the resultant acceleration of B relatively to O will be equal to the vector sum of the acceleration a_b and a_a.

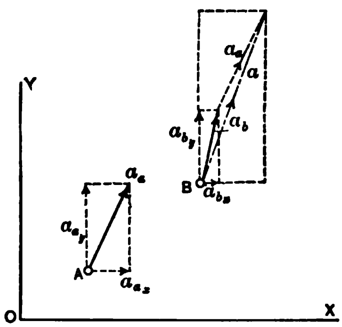

FIG. 158a.

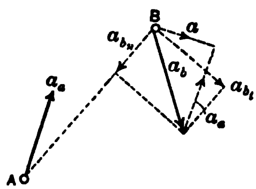

FIG. 158b.

When the particles move in the same plane the proof of this proposition is similar to that for velocities (Art. 146).

Resolving the accelerations a_b and a_a (Fig. 158a) into components parallel to OX and OY, in the same manner as the velocities v_b and v_a (Art 146) we determine that the magnitude of the resultant acceleration of B relatively to O is equal to

$$a = \sqrt{(a_{b_x} + a_{a_x})^2 + (a_{b_y} + a_{a_y})^2}, \quad . \quad . \quad . \quad . \quad (9)$$

and that a is the vector sum of a_b and a_a.

As in the case of velocities, when the distance AB is invariable, the motion of B relatively to A is circular and the acceleration a_b (Fig. 158b) may be resolved into a tangential component

$$a_{b_t} = \alpha r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and a normal component

$$a_{b_n} = \omega^2 r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where $r = AB$, α = the angular acceleration and ω = the angular velocity of B with respect to A.

Conversely, if in this case the resultant accelerations of B and A relatively to O are known, the normal and tangential components of the acceleration of B relatively to A may be determined by resolution as indicated in Fig. 158b.

148. Momentum. — The momentum of a particle of mass m, moving with a velocity v, is equal to mv (Art. 10).

According to Newton's Second Law of Motion (Art. 13) the force required to impart an acceleration a to a particle in the direction of its motion will be equal to

$$F = ma = m\frac{dv}{dt} = m\frac{d^2s}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Integrating equation (1) we obtain

$$\int_0^t F\,dt = m\int_{v_0}^{v_1} dv = m\,(v_1 - v_0), \quad . \quad . \quad . \quad . \quad (2)$$

where v_0 = the initial velocity and v_1 = the velocity at the end of the time t.

If F is a constant, equation (2) becomes

$$Ft = m\,(v_1 - v_0). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence the change in the momentum of a particle, produced by a constant force, is equal to the product of the force and the time.

The quantity $\int Fdt$ is called the *impulse of the force.*

If the force acting on the particle is variable, the expression for the impulse during the time t will be

$$\int_0^t F\,dt.$$

Hence the impulse communicated to a particle during an interval of time is equal to its change of momentum during that interval.

Equations (2) and (3) may also be said to represent the *time effect of a force.*

It is evident that the momentum of a particle is a vector quantity having direction as well as magnitude, which may therefore

be resolved into components in different directions and, conversely, if a particle has simultaneously impressed upon it momenta in different directions, the resultant momentum will be equal to the vector sum of the components.

Moreover, the vectors representing the different momenta will be in the same direction as, and proportional to, the forces required to produce the momenta in any given unit of time.

By multiplying equation (3) (Art. 146) by m we obtain the following algebraic expression for the resultant momentum of the particle in terms of its components in any two directions at right angles to each other;

$$mv = \sqrt{(mv_x)^2 + (mv_y)^2}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

149. Moment of Momentum. — *The moment of the momentum of a particle with respect to any center, or axis, is equal to the product of its momentum and the length of the perpendicular from the center, or axis, to the line of direction of its motion.*

If the particle of mass m moves in a circular path of radius r, the moment of its momentum about the center will be equal to

$$mvr = m\omega r^2,$$

where ω = the angular velocity of the particle about the center.

150. Tangential and Normal Components — Deviating Forces. — When a particle moves in a curved path, the resultant of the forces acting upon it does not act in the direction of the motion, but must act so as to deflect the particle from a straight path (Art. 141).

Curvilinear motion may be produced by a constant force, or by a force, or system of forces, varying in any manner. The form of the path will depend on the variation in magnitude and direction of the resultant force.

An important case is that in which the resultant force always acts toward a fixed point; for example, that of a body moving in an elliptical orbit under the influence of a force which always acts toward a focus. Such a force is called a *central force*, and the fixed point the *center of force.*

In any case where a particle moves in a curved path, the resultant force will act in the direction of the resultant acceleration. This force may be resolved at any instant into a component in the direction of the tangent, which will be equal to

$$F_t = ma_t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and a component in the direction of the normal, which will be equal to

$$F_n = ma_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where a_t and a_n are the tangential and normal components of the acceleration of the particle.

Substituting for a_t and a_n their values (Art. 143) we shall have

$$F_t = ma_t = m\frac{dv}{dt} = m\frac{d^2s}{dt^2}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$F_n = ma_n = m\frac{v^2}{r} = \frac{m}{r}\left(\frac{ds}{dt}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The component F_t will produce a change in speed only: it will be called for convenience the *tangential force*. Since the component F_n will produce a change in the direction of the motion only, it will be called the *deviating force*. The resultant force acting on the particle will be equal to

$$F = \sqrt{F_n^2 + F_t^2} = m\sqrt{\frac{1}{r^2}\left(\frac{ds}{dt}\right)^4 + \left(\frac{d^2s}{dt^2}\right)^2}. \quad . \quad . \quad (5)$$

Transposing the terms in equation (4), we obtain the expression for the radius of curvature of the path in terms of the deviating force and the velocity,

$$r = \frac{mv^2}{F_n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If $F_t = 0$ the particle will move with *uniform* speed in a path whose radius of curvature is everywhere inversely proportional to F_n; and, if F_n = a constant also, the path will evidently be a circle.

151. Centripetal and Centrifugal Forces. — When a particle moves in a circular path the deviating force F_n will always act toward the center and is commonly called the *centripetal force*. Its magnitude will evidently be that given by formula (4) (Art. 150).

An independent solution for the case of a particle moving with uniform speed in a circular path ABE (Fig. 159) is the following.

Let v = the velocity of the particle and let r = the radius of the circle ABE. Let A be any point in the path and let the arc

AB represent the distance through which the particle will move in any time t. The length of the arc AB will be equal to vt. This motion may be considered as made up of two components, one

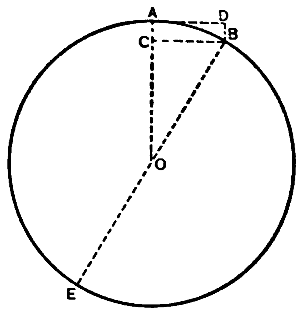

FIG. 159.

equal to AD, in the direction tangent to the path, and the other equal to $DB = AC$, in the direction parallel to AO, the radius to the point A. Let $AC = s$.

The angle AOB is equal to $\frac{vt}{r}$ and

$$DB = s = r - r\cos\frac{vt}{r}.$$

Hence the acceleration at the point B in a direction parallel to AO will be equal to

$$\frac{d^2s}{dt^2} = \frac{v^2}{r}\cos\frac{vt}{r}.$$

When $\quad t = 0, \quad \cos\frac{vt}{r} = 1 \quad$ and $\quad \frac{d^2s}{dt^2} = \frac{v^2}{r} = a_n.$

Therefore the centripetal force acting on the particle will be equal to

$$F_n = ma_n = \frac{mv^2}{r} = m\omega^2 r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ω = the angular velocity (Art. 144).

If the speed of a particle in a circular path is not uniform, equation (1) will evidently give the magnitude of the centripetal

force in terms of the velocity at any instant, and the tangential component of the force acting on the particle will be equal to

$$F_t = ma_t = m\alpha r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where α = the angular acceleration at the given instant. Hence the resultant force will in this case be equal to

$$F = \sqrt{F_t^2 + F_n^2} = m\sqrt{a_t^2 + a_n^2} = mr\sqrt{\alpha^2 + \omega^4}. \quad . \quad (3)$$

According to the law of action and reaction, in every case of curvilinear motion there must be a force exerted through the center of curvature which is equal and opposite to the deviating force acting on the particle.

When the particle moves in a circular path, this *reaction outward*, on the axis through the center of the circle, is commonly called the *centrifugal force.*

152. Work and Energy. — Whenever a force produces motion in its point of application, *work* is performed. The work done by a constant force F in moving its point of application through a distance s, in the direction of its line of action, is equal to

$$Fs.$$

If the force is not constant the work done will be equal to

$$\int_0^s F\,ds.$$

In mechanics, energy may be defined as the capacity for doing work which is possessed by a system of bodies, or particles, in virtue of their velocities, or relative positions. The energy due to velocity is called *kinetic energy* and that due to position is called *potential energy.*

In accordance with the above definition, the *kinetic energy* of a body will be equal to the work necessary to produce its velocity. In the case of a freely moving particle of mass m, which is acted upon by a force F during a change in its velocity from 0 to v, we have

$$F\,dt = m\,dv \text{ (Art. 146)},$$

which may be written

$$Fv\,dt = mv\,dv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Putting $ds = v\,dt$ and integrating, we have

$$\int_0^s F\,ds = \int_0^v mv\,dv = \tfrac{1}{2}\,mv^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But $\int_0^s F\,ds$ is the work done in producing the velocity v and hence the kinetic energy of the particle will be equal to

$$E = \tfrac{1}{2}\,mv^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

By integrating equation (1) between the limits $v = v_1$, when $s = s_1$, and $v = v_0$, when $s = s_0$, we obtain

$$\int_{s_0}^{s_1} F\,ds = \tfrac{1}{2}\,m\,(v_1^2 - v_0^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which may be written

$$\int_{s_0}^{s_1} F\,ds = E_1 - E_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where E_0 and E_1 are the initial and final values of the kinetic energy.

Hence the change in kinetic energy during any displacement of the particle is equal to the work done by the impressed force. The above equation may be said to represent the *space effect of a force* acting on a particle.

153. Motion of a Particle under the Action of the Force of Gravity. — The value of g, the downward acceleration due to gravity at different points on the Earth's surface, as expressed by the formula (Art. 18), may be considered to be constant for comparatively small heights. Hence, if we neglect the resistance of the air, or assume that the particle falls in a vacuum, and let h equal the distance through which it moves in the time t, we have, by substituting in equations (2) and (3) (Art. 143),

$$v = v_0 + gt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$h = v_0 t + \frac{1}{2}gt^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where v_0 is the initial velocity downward.

If the initial velocity is upward, we may consider the acceleration g to be a negative quantity, in which case

$$v = v_0 - gt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$h = v_0 t - \frac{1}{2}gt^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where v is the velocity and h the height of the particle above the starting point at the end of the time t.

If the initial velocity is equal to zero, we have the case of a particle falling freely from rest, and

$$v = gt, \qquad (5)$$

$$h = \frac{1}{2} gt^2. \qquad (6)$$

Eliminating t between equations (5) and (6),

$$v = \sqrt{2\,gh}, \qquad (7)$$

or

$$h = \frac{v^2}{2\,g}, \qquad (8)$$

where h is the height through which the particle, starting from rest, must fall to acquire the velocity v.

If we eliminate t between equations (1) and (2) we shall obtain

$$v^2 - v_0^2 = 2\,gh, \qquad (9)$$

and

$$h = \frac{v^2 - v_0^2}{2\,g}, \qquad (10)$$

which gives the distance through which the particle must fall to acquire the velocity v, when its initial velocity is v_0.

154. Motion of a Particle on a Frictionless Inclined Plane under the Action of Gravity. — Let θ equal the angle which the plane makes with the horizontal. The forces acting on the particle at any instant will be the force w, exerted by gravity, and the normal pressure N, exerted by the plane (Fig. 160). The resultant force acting on the particle will be equal to

$$F = w \sin \theta.$$

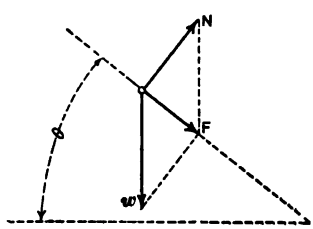

FIG. 160.

If the mass of the particle is equal to $m = \dfrac{w}{g}$, its acceleration will be equal to

$$a = \frac{F}{m} = g \sin \theta.$$

Substituting this value in equations (2) and (3) (Art. 143) we have, when the initial velocity is downward,

$$v = v_0 + g \sin \theta t, \quad \ldots \ldots \quad (1)$$

$$s = v_0 t + \frac{1}{2} g \sin \theta t^2. \quad \ldots \ldots \quad (2)$$

When the initial velocity is upward, the value of g, in equations (1) and (2), may be considered to be negative.

When the initial velocity is zero we obtain, by eliminating t,

$$v = \sqrt{2\, gs \sin \theta} = \sqrt{2\, gh}, \quad \ldots \ldots \quad (3)$$

where $h = s \sin \theta$, the vertical projection of the distance through which the particle moves.

When the initial velocity is v_0, if we eliminate t between equations (1) and (2), we shall obtain

$$v^2 - v_0^2 = 2\, gs \sin \theta = 2\, gh. \quad \ldots \ldots \quad (4)$$

Therefore, when a particle slides down a frictionless inclined plane under the action of gravity, through a vertical distance h, the velocity of the particle in its path will be the same as that which it would acquire in falling vertically through the same height.

155. Motion of a Particle along a Curve under the Action of Gravity. — We will consider only the case where the path is a

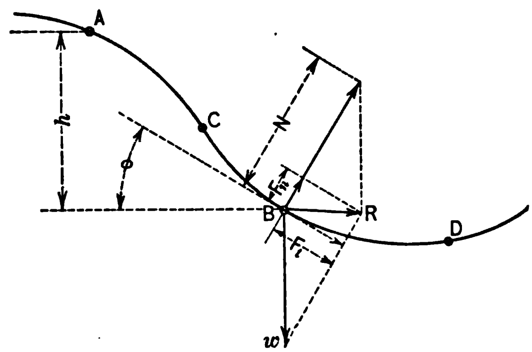

Fig. 161.

curve situated in a vertical plane and there is no friction between the particle and the surface on which it slides. Let the particle start from rest at A and slide down the curved surface ACD (Fig. 161).

When the particle is at any point B the forces acting upon it

will be w, the force exerted by gravity, and the normal pressure N, exerted by the surface.

The resultant force R will be the vector sum of w and N.

(*a*) *Velocity.* — Imagine the path to consist of a large number of short, straight lines tangent to the curve. The velocity acquired by the particle in sliding down each of the straight lines is the same as that which it would acquire in falling through a distance equal to the vertical projection of that line (Art. 154), and hence the velocity acquired in starting from rest and sliding down the broken line through a vertical distance h will be equal to $\sqrt{2\,gh}$.

Since the curve is the limiting form of the path obtained by increasing the number of the tangent lines indefinitely, it is evident that the velocity acquired by a particle, starting from rest at A and sliding down the curve through the distance AB, will also be equal to

$$v = \sqrt{2\,gh}, \quad \ldots \ldots \ldots \quad (1)$$

where h is the vertical projection of AB.

If the particle has an initial velocity v_0 at A and its velocity at B is equal to v, it may easily be shown in a manner similar to the above that

$$v^2 - v_0^2 = 2\,gh. \quad \ldots \ldots \ldots \quad (2)$$

(*b*) *Normal and Tangential Components.* — The resultant force R, acting on the particle at any point B in its path, may be resolved into a tangential component

$$F_t = m\frac{d^2s}{dt^2}, \quad \ldots \ldots \ldots \quad (3)$$

and a normal component

$$F_n = m\frac{v^2}{r} \text{ (Art. 150).}$$

If $v_0 = 0$,

$$v = \sqrt{2\,gh},$$

and

$$F_n = \frac{w}{g}\cdot\frac{2\,gh}{r} = \frac{2\,wh}{r}, \quad \ldots \ldots \quad (4)$$

where r = the radius of curvature of the path at B.

An inspection of the force parallelograms drawn at the point B will show that, if we let ϕ equal the angle between the tangent to the curve at B and the horizontal,

$$F_t = w \sin\phi, \quad \ldots \ldots \ldots \quad (5)$$

and

$$N = w\cos\phi + F_n = w\left(\cos\phi + \frac{2\,h}{r}\right). \quad \ldots \quad (6)$$

It is evident that at the point of inflexion C, $F_n = 0$, and that at the point D, where the tangent is horizontal, $F_t = 0$.

(c) *Acceleration.* — The tangential acceleration at any point in the path will be equal to

$$a_t = \frac{d^2s}{dt^2} = \frac{F_t}{m} = g \sin \phi, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and the normal acceleration,

$$a_n = \frac{v^2}{r} = \frac{2\,gh}{r}.$$

(d) *Time of Descent.* — If the particle starts from rest, the time of descent from A to B will be equal to

$$t = \int \frac{ds}{v} = \int \frac{ds}{\sqrt{2\,gh}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and will evidently depend on the form of the path.

156. Simple Circular Pendulum. — Time of Oscillation. — Since the simple circular pendulum is a particle moving in the arc of a vertical circle under the action of gravity, the formulas in Art. 155 will apply in this case.

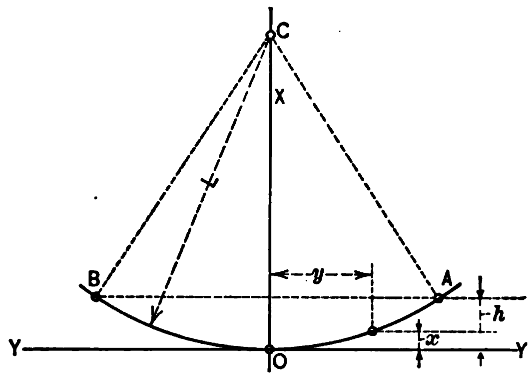

FIG. 162.

Let AOB (Fig. 162) be the path, the center being at C. Refer the motion to the vertical and horizontal axes, OX and OY, through the lowest point in the path. Let L = the length of the pendulum and h = the distance of A, the highest point in its path from the axis OY.

Then the equation for the time of descent from A to any point in the path whose coördinates are (x, y) will be

$$t = \int_{x=0}^{x=h} \frac{ds}{\sqrt{2g(h-x)}} \quad \text{(Art. 155).}$$

Expressing ds in terms of dx and dy, we obtain for the time of a single oscillation from A to B

$$t = 2\int_{x=0}^{x=h} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{2g(h-x)}}\,dx. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

From the equation of the path,

$$y^2 = 2Lx - x^2,$$

we obtain

$$\frac{dy}{dx} = \frac{L-x}{y},$$

and

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{L^2}{2Lx - x^2} = \frac{L}{x}\left(\frac{L}{2L-x}\right) = \frac{L}{2x}\left(1 - \frac{x}{2L}\right)^{-1}.$$

Substituting this value in equation (1) and reducing, we have

$$t = \sqrt{\frac{L}{g}}\int_0^h \left(1 - \frac{x}{2L}\right)^{-\frac{1}{2}} \frac{dx}{\sqrt{hx - x^2}}. \quad . \quad . \quad . \quad (2)$$

This equation can be integrated approximately by expanding the binomial

$$\left(1 - \frac{x}{2L}\right)^{-\frac{1}{2}},$$

which will give

$$t = \sqrt{\frac{L}{g}}\int_0^h \left(1 + \frac{x}{4L} + \frac{3x^2}{32L^2} + \cdots\right)\frac{dx}{\sqrt{hx - x^2}}. \quad . \quad (3)$$

Integrating equation (3) and omitting all the terms in the series containing powers of x above the first we shall have

$$t = \sqrt{\frac{L}{g}}\left[\text{versin}^{-1}\frac{2x}{h} + \frac{1}{4L}\left(\frac{h}{2}\,\text{versin}^{-1}\frac{2x}{\text{h}} - \sqrt{hx - x^2}\right)\right]_0^h$$

$$= \pi\sqrt{\frac{L}{g}}\left(1 + \frac{h}{8L}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which gives an approximate value for t, the error depending upon the magnitude of h.

If h is small, compared with L, the last term may be omitted, giving the formula

$$t = \pi\sqrt{\frac{L}{g}}, \quad \ldots \ldots \quad (5)$$

which is the one most commonly used.

The maximum allowable value of h for any given degree of accuracy when equation (5) is used may be very closely determined from equation (4).

157. Simple Cycloidal Pendulum. — Assume that the path AOB (Fig. 162) is the arc of a cycloid. If r equals the radius of the describing circle, the equation of the cycloid, referred to the axes OX and OY, will be

$$y = r \operatorname{versin}^{-1}\frac{x}{r} + \sqrt{2rx - x^2},$$

from which
$$\frac{dy}{dx} = \sqrt{\frac{2r - x}{x}},$$

and
$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{2r}{x}.$$

Substituting this value in equation (1) (Art. 156) we obtain

$$t = 2\sqrt{\frac{r}{g}}\int_0^h \frac{dx}{\sqrt{hx - x^2}}. \quad \ldots \ldots \quad (1)$$

Integrating
$$t = 2\sqrt{\frac{r}{g}}\left[\operatorname{versin}^{-1}\frac{2x}{h}\right]_0^h = 2\pi\sqrt{\frac{r}{g}}. \quad \ldots \quad (2)$$

This expression is independent of h, and therefore the time of oscillation is independent of the magnitude of the arc.

158. Unresisted Projectile. — The following will apply to the case of a projectile which, after having been impressed with an initial velocity in a given direction, moves under the action of the force of gravity only, the resistance of the air being neglected.

We will consider the projectile to be a particle, situated at O (Fig. 163), to which an initial velocity is imparted in the direction OA. Let v_0 = the initial velocity, θ = angle which OA makes with the horizontal, w = the weight and m = the mass of the projectile.

Assume the horizontal and vertical axes OX and OY through the point O, and let OBC be the path followed by the projectile.

Let a = the resultant acceleration and v = the resultant velocity of the particle at any point B, whose coördinates are (x, y), and

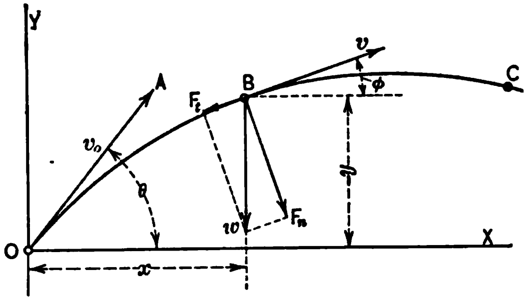

FIG. 163.

resolve a and v into components a_x, a_y and v_x, v_y, parallel to OX and OY. Since w is the only force acting on the particle,

$$a_x = \frac{d^2x}{dt^2} = 0, \quad \text{. . . . .} \quad (1)$$

and

$$a_y = \frac{d^2y}{dt^2} = -\frac{w}{m} = -g. \quad \text{. . . . . . .} \quad (2)$$

Integrating equations (1) and (2) and noting that when

$$t = 0, \; v_x = v_0 \cos\theta \;\text{ and }\; v_y = v_0 \sin\theta,$$

we obtain

$$v_x = \frac{dx}{dt} = v_0 \cos\theta, \quad \text{. . . . .} \quad (3)$$

and

$$v_y = \frac{dy}{dt} = v_0 \sin\theta - gt. \quad \text{. . . . .} \quad (4)$$

Integrating equations (3) and (4) we obtain

$$x = v_0 \cos\theta\, t, \quad \text{. . . . .} \quad (5)$$

and

$$y = v_0 \sin\theta\, t - \frac{1}{2} gt^2. \quad \text{. . . . . .} \quad (6)$$

By eliminating t between equations (5) and (6) we obtain the equation of the path of the projectile,

$$y = x \tan\theta - \frac{gx^2}{2\, v_0{}^2 \cos^2\theta}, \quad \text{. . . . . .} \quad (7)$$

which is evidently a parabola.

The resultant velocity at B will be equal to

$$v = \sqrt{(v_x)^2 + (v_y)^2}, \quad \text{. . . . .} \quad (8)$$

and if ϕ = the angle between the tangent at B and the horizontal,

$$\tan \phi = \frac{v_y}{v_x} = \frac{dy}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The magnitude of the resultant acceleration at B will be equal to

$$a = \sqrt{(a_x)^2 + (a_y)^2} = g, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

in the direction of the resultant force w.

Resolving a into its normal and tangential components (Art. 143) we obtain

$$a_t = g \sin \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$a_n = g \cos \phi = \frac{v^2}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

From equation (12) we obtain the expression

$$r = \frac{v^2}{g \cos \phi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

which gives the length of the radius of curvature of the path at B. If we resolve the resultant force w into its normal and tangential components, F_n and F_t, we shall have

$$F_n = w \cos \phi = \frac{mv^2}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and

$$F_t = w \sin \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

It is evident that, if we should assume a constant retarding force F, acting in the horizontal direction, equations (1) and (2) would be written $a_x = -\frac{F}{m}$ and $a_y = -g$. From these, the equation of the path followed by the particle, and the components of its velocity and acceleration at any point, could be determined in the same manner as the above.

159. Harmonic Motion. — A particle moving in a straight line with an acceleration which is always directed towards a point on the line and proportional to its distance from that point is said to have a *simple harmonic motion.*

If we assume the line as the axis of X and the point O, towards which the acceleration is directed, as the origin, the acceleration of the particle when at a distance x from O will be equal to

$$a = \frac{d^2x}{dt^2} = -cx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where c is a positive constant; the negative sign indicating that when x is positive the acceleration is in the negative direction, and vice versa.

If we let m = the mass of the particle, the force required to impart the acceleration at any instant will be equal to

$$F = -mcx, \quad \ldots \ldots \quad (2)$$

and its magnitude will be proportional to the distance x.

It will be seen from the following that if a point C (Fig. 164a) moves with a uniform speed in a circle, the motion of a particle at D, the projection of C on a diameter, will fulfill the above conditions.

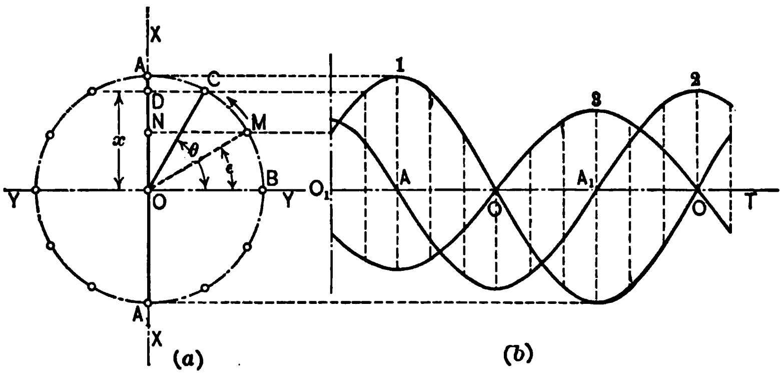

FIG. 164.

Let r = the radius of the circle, ω = the angular velocity of the point C, and t = the time elapsing while C moves over the arc BC.

Then $$\theta = \omega t$$

and $$x = r \sin \omega t. \quad \ldots \ldots \quad (3)$$

Hence,

$$v = \frac{dx}{dt} = \omega r \cos \omega t \quad \ldots \ldots \quad (4)$$

$V = \omega\sqrt{r^2 - y^2}$

and

$$a = \frac{d^2x}{dt^2} = -\omega^2 r \sin \omega t = -\omega^2 x. \quad \ldots \ldots \quad (5)$$

Therefore, the motion of the point D is harmonic and the constant c (equation 1) is equal to ω^2. Neglecting signs, the magnitude of the force acting on the particle will be equal to

$$F = m\omega^2 x, \quad \ldots \ldots \quad (6)$$

the force always acting towards O and being directly proportional to x. Such a motion is evidently periodic, the same cycle being repeated for each revolution of the point C.

When the particle is at O,

$$F = 0; \quad \ldots \ldots \ldots \quad (7)$$

and when the particle is at A or A_1,

$$F = m\omega^2 r, \quad \ldots \ldots \ldots \quad (8)$$

its magnitude being the same as that of the centripetal force acting on a particle of equal mass m, moving with the point C.

The time T of a complete oscillation of the point D is called the *period* of the motion. Since this will be the same as the time of a complete revolution of the point C,

$$T = \frac{2\pi}{\omega} \quad \ldots \ldots \ldots \quad (9)$$

and, since the magnitude of the acceleration is always $a = \omega^2 x$, we may write

$$T = 2\pi\sqrt{\frac{x}{a}}. \quad \ldots \ldots \ldots \quad (10)$$

By substituting the value of ω obtained from equation (8), we also have

$$T = 2\pi\sqrt{\frac{mr}{F}}. \quad \ldots \ldots \ldots \quad (11)$$

The reciprocal of the period, namely,

$$n = \frac{1}{T} = \frac{\omega}{2\pi}, \quad \ldots \ldots \ldots \quad (12)$$

is called the *frequency*, and is evidently equal to the number of oscillations per unit of time. The distance $OA = r$ is called the *amplitude* of the motion.

If the time is reckoned from some point other than O, such as N, the projection of the point M on the diameter,

$$\theta = \omega t + \epsilon,$$

where the angle ϵ is equal to the *lead*, when positive, or *lag*, when negative, and is sometimes called the *epoch angle*, or *epoch*, of the motion. The angle $\omega t + \epsilon$ is called the *phase angle* of the motion.

Substituting the above value of θ in equation (3) and differentiating as before, we obtain

$$x = r \sin (\omega t + \epsilon), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$
$$v = \omega r \cos (\omega t + \epsilon), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$
$$a = - \omega^2 r \sin (\omega t + \epsilon) = - \omega^2 x. \quad . \quad . \quad . \quad . \quad (15)$$

The characteristics of the motion are shown by space-time, velocity-time, and acceleration-time plots marked respectively (1), (2) and (3) (Fig. 164b).

The space-time plot (1) may be constructed by dividing the base O_1T and the circumference of the circle into the same number of equal parts and projecting the points on the circumference on the ordinates erected at the corresponding points on O_1T. The curve drawn through the extremities of the ordinates will give the displacement of the point D after any interval of time, and will evidently be a sinusoid.

In constructing the velocity-time plot (2) it may be noted that equation (14) can be written

$$v = \omega y,$$

where $y = r \cos (\omega t + \epsilon)$ is the coördinate of the point C at any instant. Hence the ordinates of the curve may be determined by multiplying the y coördinates of the successive positions of the point C by the constant ω.

The ordinates of the acceleration-time plot (3) may evidently be determined by multiplying the ordinates of the space-time plot by $- \omega^2$.

A familiar example of a body having a simple harmonic motion is that of the slotted crosshead driven by a crank turning at a uniform speed. Another example is that of a weight on a vibrating spring, in which case the accelerating force will vary directly as the distance of the weight from the center of its motion.

The motion of the simple pendulum (Art. 156) is also approximately harmonic, for if we let θ = the angle between OC and the radius from the center C to any point (x, y) in the path (Fig. 162), we shall have $y = L \sin \theta$ and the tangential acceleration at the point will be equal to $a_t = g \sin \theta$ (Art. 155).

Hence $\dfrac{y}{a_t} = \dfrac{L}{g}$ = a constant, and, since for small amplitudes the path is nearly a straight line, the motion is very nearly harmonic.

We may, therefore, substitute $\frac{x}{a} = \frac{L}{g}$ in equation (10) thus obtaining the formula for the approximate period of oscillation of the pendulum,

$$T = 2\pi\sqrt{\frac{L}{g}}. \quad \ldots \ldots \ldots \quad (16)$$

The motion of the crosshead of the ordinary reciprocating engine is approximately harmonic, the degree of approximation depending on the ratio of the length of the crank and connecting rod.

160. Composition and Resolution of Simple Harmonic Motions. —

CASE I. — When a point A has a simple harmonic motion with respect to a point B and the point B has in turn a simple harmonic motion with respect to a fixed point O, both motions being in the same straight line and of the same period T, the resultant motion of A with respect to O is harmonic.

Since both motions have the same period, the value of ω will be the same for each, and, if we let r_1 = the amplitude and ϵ_1 = the epoch of the first motion and r_2 = the amplitude and ϵ_2 = the epoch of the second, the displacement of A with respect to O at the end of any time t will be equal to

$$x = r_1 \sin(\omega t + \epsilon_1) + r_2 \sin(\omega t + \epsilon_2), \quad \ldots \quad (1)$$

which may be easily transformed into

$$x = (r_1 \cos \epsilon_1 + r_2 \cos \epsilon_2) \sin \omega t + (r_1 \sin \epsilon_1 + r_2 \sin \epsilon_2) \cos \omega t. \quad (2)$$

Substituting for the constants in the parentheses,

$$r_1 \cos \epsilon_1 + r_2 \cos \epsilon_2 = r \cos \epsilon \quad \ldots \ldots \quad (3)$$

and

$$r_1 \sin \epsilon_1 + r_2 \sin \epsilon_2 = r \sin \epsilon, \quad \ldots \ldots \quad (4)$$

equation (2) reduces to the form

$$x = r \cos \epsilon \sin \omega t + r \sin \epsilon \cos \omega t = r \sin(\omega t + \epsilon), \quad . \quad (5)$$

which shows that the displacement is a similar function of the time to that in Art. 159 and hence the resultant motion of A with respect to O is a simple harmonic motion.

By squaring equations (3) and (4) and adding together we may determine the value of the amplitude r of the resultant motion, in terms of r_1, r_2, ϵ_1 and ϵ_2, as follows:

$$r^2 (\cos^2 \epsilon + \sin^2 \epsilon) = (r_1 \cos \epsilon_1 + r_2 \cos \epsilon_2)^2 + (r_1 \sin \epsilon_1 + r_2 \sin \epsilon_2)^2,$$

which easily reduces to

$$r^2 = r_1^2 + r_2^2 + 2\,r_1r_2 \cos(\epsilon_2 - \epsilon_1). \quad . \quad . \quad . \quad . \quad (6)$$

Dividing equation (4) by equation (3) we obtain

$$\tan \epsilon = \frac{r_1 \sin \epsilon_1 + r_2 \sin \epsilon_2}{r_1 \cos \epsilon_1 + r_2 \cos \epsilon_2}, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which gives the epoch angle of the resultant motion.

The velocity and acceleration of the resultant motion can evidently be determined by substituting the values of r and ϵ in equations (14) and (15), (Art. 159). The resultant velocity and acceleration might also be determined by the method of composition given in Arts. 146 and 147.

A geometrical illustration of the above is shown in Fig. 165 where B is the projection of the point C_1 which revolves with the angular velocity ω about the fixed point O, and A is the projection of the point C_2, which revolves with the same angular velocity ω about the moving point B. It is evident from the construction that the angle between the radii r_1 and r_2 will be constant and equal to $\epsilon_2 - \epsilon_1$ for all positions of A and B and that, if we draw OE parallel to r_2, the diagonal r, of the parallelogram constructed with OE and r_1 as sides, will be the radius to the point C, which, when revolving with an angular velocity ω about O, will always have the point A for its projection on OX. Hence r will equal the amplitude of the resultant motion.

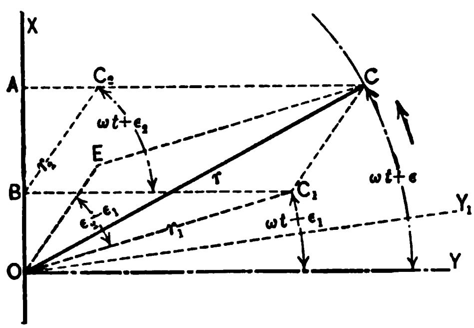

FIG. 165.

The angle $\epsilon_2 - \epsilon_1$ will be equal to the difference in phase of the two motions and, if we draw OY_1, making the angle $YOY_1 = \omega t$,

the epoch angles of the two components and the resultant motion will be respectively equal to

$$\epsilon_1 = Y_1OC_1,$$
$$\epsilon_2 = Y_1OE,$$
and
$$\epsilon = Y_1OC.$$

Hence the resultant of two simple harmonic motions of the same period T, in the same line, will be a simple harmonic motion of the period T whose amplitude r is equal to the vector sum of the amplitudes of the component motions, the directions of the vectors being determined by the difference in phase of the components.

It evidently follows that the resultant of any number of simple harmonic motions of the same period T, in the same line, will be a simple harmonic motion of period T; since two of the motions

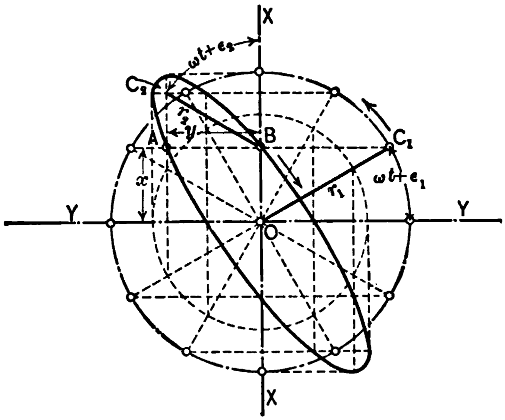

Fig. 165a.

may be combined into a simple harmonic motion as above, and this resultant motion with a third and so on until a simple harmonic motion of period T is obtained for the resultant. The amplitude of the resultant motion may be obtained by laying off the vectors, representing the amplitudes of the separate motions, as the sides of a polygon, the external angles of which are equal to the successive differences in phase.

Conversely, a simple harmonic motion may be resolved into two or more simple harmonic motions in the same line by the reverse of the above process.

Two simple harmonic motions of different periods, in the same line, may be compounded in the same manner as the above but in this case the angle C_1OE and hence the radius $r = OC$ will vary and the resulting motion will not be harmonic.

CASE II. — When a point A has a simple harmonic motion with respect to a point B and the point B has in turn a simple harmonic motion with respect to a fixed point O, the two motions being in directions at right angles and having the same period T, the resultant motion of A with respect to O may be determined as follows:

Assume the coördinate axes OX and OY, with OX coinciding with the line of motion of the point B (Fig. 165a). The coördinates of the point A after any interval of time t will be equal to

$$x = r_1 \sin(\omega t + \epsilon_1),$$
$$y = r_2 \sin(\omega t + \epsilon_2).$$

By plotting the successive positions of A, for any values of r_1, ϵ_1, r_2 and ϵ_2, its path may be determined graphically in a manner similar to that in the preceding case.

The equation of the path may be found by eliminating t between the above equations as follows:

$$\sin^{-1}\frac{x}{r_1} = \omega t + \epsilon_1,$$

$$\sin^{-1}\frac{y}{r_2} = \omega t + \epsilon_2,$$

and hence

$$\sin^{-1}\frac{y}{r_2} - \sin^{-1}\frac{x}{r_1} = \epsilon_2 - \epsilon_1;$$

and by transposing we obtain

$$\frac{y}{r_2} = \sin\left[\sin^{-1}\frac{x}{r_1} + (\epsilon_2 - \epsilon_1)\right]. \quad . \quad . \quad . \quad . \quad (1)$$

Expanding equation (1),

$$\frac{y}{r_2} = \sin\left(\sin^{-1}\frac{x}{r_1}\right)\cos(\epsilon_2 - \epsilon_1) + \cos\left(\sin^{-1}\frac{x}{r_1}\right)\sin(\epsilon_2 - \epsilon_1),$$

and substituting

$$\cos\sin^{-1}\frac{x}{r_1} = \frac{\sqrt{r_1^2 - x^2}}{r_1},$$

$$\frac{y}{r_2} = \frac{x}{r_1}\cos(\epsilon_2 - \epsilon_1) + \frac{\sqrt{r_1^2 - x^2}}{r_1}\sin(\epsilon_2 - \epsilon_1).$$

Transposing and squaring to eliminate the radical, we obtain

$$\frac{y^2}{r_2^2} - 2\frac{yx}{r_2r_1}\cos(\epsilon_2 - \epsilon_1) + \frac{x^2}{r_1^2}\cos^2(\epsilon_2 - \epsilon_1)$$
$$= \frac{r_1^2 - x^2}{r_1^2}\sin^2(\epsilon_2 - \epsilon_1);$$

and hence

$$\frac{y^2}{r_2^2} - \frac{2\,yx}{r_1r_2}\cos(\epsilon_2 - \epsilon_1) + \frac{x^2}{r_1^2}[1 - \sin^2(\epsilon_2 - \epsilon_1)]$$
$$- \frac{r_1^2 - x^2}{r_1^2}\sin^2(\epsilon_2 - \epsilon_1) = 0;$$

which easily reduces to

$$\frac{y^2}{r_2^2} + \frac{x^2}{r_1^2} - \frac{2\,xy}{r_1r_2}\cos(\epsilon_2 - \epsilon_1) = \sin^2(\epsilon_2 - \epsilon_1). \quad . \quad . \quad (2)$$

As in the preceding case, the difference of phase, $\epsilon_2 - \epsilon_1$, will be a constant, and hence equation (2) is the equation of an ellipse referred to the axes OX and OY (Fig. 165a). The resultant motion of A is in this case called *elliptic harmonic motion*, and the period is the same as that of the component motions.

The resultant velocity and resultant acceleration of A at any point in its path may be determined by the methods of composition in Arts. 146 and 147.

Special Cases.— (IIa) When the difference of phase, $\epsilon_2 - \epsilon_1 = 0$, equation (2) becomes

$$\left(\frac{x}{r_1} - \frac{y}{r_2}\right)^2 = 0,$$

and when $\epsilon_2 - \epsilon_1 = \pi$, equation (2) becomes

$$\left(\frac{x}{r_1} + \frac{y}{r_2}\right)^2 = 0,$$

the motion in either case being in a straight line and simple harmonic.

(IIb) When the difference of phase,

$$\epsilon_2 - \epsilon_1 = \frac{\pi}{2},$$

equation (2) becomes

$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 1,$$

and the path is an ellipse whose semi-major and semi-minor axes are equal to the amplitudes r_1 and r_2 and coincide in direction with OX and OY.

(IIc) When $\epsilon_2 - \epsilon_1 = \frac{\pi}{2}$ and the amplitudes $r_1 = r_2 = r$, equation (2) becomes $x^2 + y^2 = r^2$ and the particle moves with uniform motion in a circle.

When the periods of the component motions are different, the path of the point A may be plotted in the manner indicated, but the motion, except in cases where the ratio of the periods is a rational number, will not be periodic.

161. Problems. — Kinetics of the Particle. — In the solution of the following problems, the mass in each case may be treated as though it were a single particle and the forces acting upon it as if applied at the same point. It will be apparent from a later study of the kinetics of rigid bodies that in most of the solutions no error will be introduced, and in others only a slight one, by imposing these conditions. Unless otherwise stated, the resistance to motion exerted by the air will be neglected in each case.

Problem 1.

A weight of 5 lbs. is thrown vertically upward with an initial velocity of 40 ft. per sec. How far will it rise? What time will elapse before it again reaches its starting point? What will be its velocity when it has reached the height of 20 ft.? What will be its velocity when it has reached the highest point? What will be its acceleration at that point?

Problem 2.

A body is thrown vertically downward from a height of 50 ft. above the ground with an initial velocity of 20 ft. per sec. What will be its velocity when it strikes the ground? What will be its acceleration at that point?

Problem 3.

The four weights A, B, C and D, attached to a flexible cord (Fig. 166), are drawn along a horizontal plane by a force F. If $F = 20$ lbs. and the weight

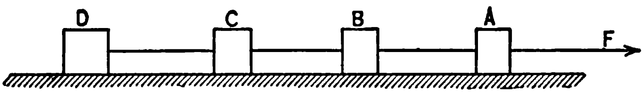

FIG. 166.

$A = 10$ lbs., $B = 20$ lbs., $C = 15$ lbs. and $D = 30$ lbs., determine the following, neglecting the weight of the cord and friction:

(a) The acceleration of the weights.

(b) The tension in the sections AB, BC and CD of the cord.

(c) The work done in moving each weight through a distance of 10 ft.

(d) The velocity of the weights after moving 15 ft. from the position of rest.

Problem 4.

Solve Problem 3, assuming a constant resistance due to friction between the weights and the plane, distributed as follows: $A = 1$ lb., $B = 2$ lbs., $C = 1.5$ lbs. and $D = 3$ lbs.

Problem 5.

The weights in the Atwood's machine (Fig. 167) are $A = 10$ oz., $B = 15$ oz. Find the acceleration of the weights and the tension in the cord, neglecting friction and weight of the pulley and cord. Find the velocity after 5 secs. if the initial velocity is equal to zero.

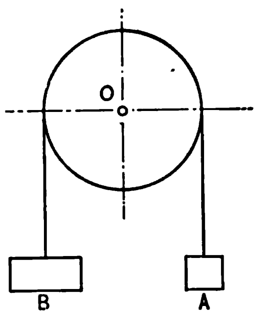

FIG. 167.

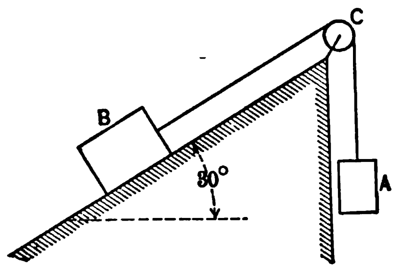

FIG. 168.

Problem 6.

Assuming the initial velocity equal to zero, find the weight necessary to add to the weight A (Prob. 5) in order to produce a velocity of 10 ft. per sec. in 5 secs. (a) in the same direction; (b) in the opposite direction.

Problem 7.

Two equal weights A and B (Fig. 168) are connected by a flexible cord running over a pulley at C. Find the acceleration of the weights and the tension in the cord, assuming no friction and neglecting the weight of the pulley and cord.

Problem 8.

If another pulley C is suspended from the cord on the Atwood's machine (Prob. 5) in place of the weight B and two weights D and E, weighing 6 oz. and 4 oz. respectively, are suspended from a cord running over C, find the accelerations of D, E and A relatively to O; also the tensions in both cords. Neglect friction and the weights of both the pulleys and cords.

Problem 9.

A weight of 12 lbs., starting from rest, is moved along a horizontal plane by a force F which varies in such a manner that

$$F = 2 + t + \frac{t^2}{10}.$$

Find the acceleration after 5 secs. Find the distance through which the weight will move in 10 secs. Find the velocity after 10 secs.

Problem 10.

Solve Problem 9, assuming that the friction between the weight and the plane is constant and equal to 1.5 lbs. If the force ceased to act at the end of 10 secs. how far would the weight move before coming to rest?

Problem 11.

Plot the space-time curve (Art. 142) and the velocity-time curve (Art. 143) for Problem 10.

Problem 12.

A body moves in a straight line in such a manner that the velocity-time curve is that shown in Fig. 169, OAB and BCD being arcs of circles. Scales:

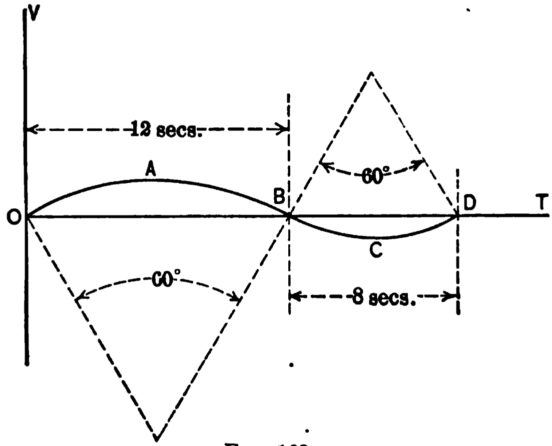

FIG. 169.

Abs. 1″ = 2 secs., Ords. 1″ = 10 ft. per sec. Plot the space-time curve. Scales: Abs. 1″ = 2 secs., Ords. 1″ = 30 ft. Solve graphically, assuming ordinates at one-second intervals. How far from the starting point will the body be at the end of 10 secs.? At the end of 20 secs.? At what point in the path will the velocity be equal to zero? At what points in the path will the resultant force acting on the body equal zero? Find the velocity and the acceleration at the end of 8 secs.; at the end of 14 secs.

Problem 13.

A body weighing 10 lbs., starting from rest, is moved along a frictionless horizontal plane by a force which is equal to

$$F = 20 + t^2 - \frac{t^3}{10}.$$

Plot the velocity-time curve and the space-time curve for values of t from 0 to 20 secs. From the plots determine the following points: (a) at which the velocity is equal to zero; (b) at which the acceleration is equal to zero; (c) at which the greatest velocity occurs; (d) at which the greatest acceleration occurs.

Problem 14.

A weight of 8 lbs., starting from rest, is drawn along a horizontal plane by a force which varies in such a manner that

$$s = 4t^2 - \frac{t^3}{5}.$$

Find the magnitude of the force acting and the velocity of the weight after 10 secs. Find the time at which the velocity is equal to zero; also, at which the acceleration is equal to zero.

Problem 15.

The weights A, B and C (Fig. 170), weighing 20 lbs., 15 lbs. and 10 lbs., respectively, attached to a slender rod OC, rotate about the center O with a uniform speed of 150 revolutions per minute. Find the tension in the sections OA, AB and BC, neglecting the weight of the rod.

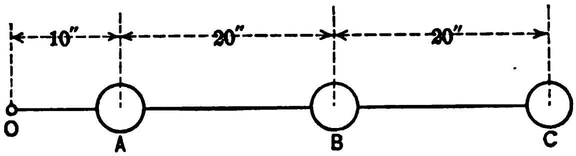

Fig. 170.

Problem 16.

Find the moment of the momentum about the axis of rotation of each of the weights and also of the entire system in Problem 15.

Problem 17.

The weight A (Fig. 171) revolves as a conical pendulum about the axis OB with a speed of 60 revolutions per minute. Find the angle ϕ, neglecting the weight of the string OA. Find the tension in OA.

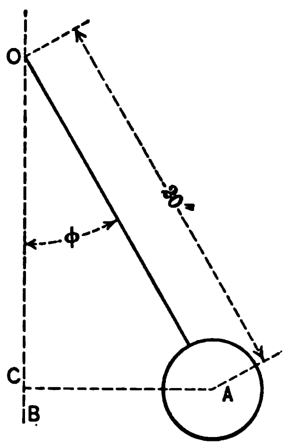

Fig. 171.

Problem 18.

Prove that for any given angular velocity ω of the weight A (Prob. 17), the vertical projection OC of the length OA (Fig. 171) will be the same for all values of OA, provided $\omega^2 (OA) > g$.

Problem 19.

A particle of weight w is attached to the rim of a wheel 10 ft. in diameter rotating on a fixed axis through its center. If the wheel starts from rest and an angular acceleration of 100 rads. per sec.^{2} is imparted to it, find the magnitude and direction, with respect to the rim, of the resultant force acting on the particle at the end of 10 secs.; also the moment of the momentum of the particle about the center of the wheel at that instant.

Problem 20.

A projectile is thrown with an initial velocity of 100 ft. per sec. at an angle of 30° above the horizontal. How high will it rise? How far will it travel in the horizontal direction before striking the ground at the same level at which it started? What time will elapse before it strikes the ground? Find the radius of curvature of the path at the highest point.

Problem 21.

Find the velocity of the projectile (Prob. 20) when it has traveled one quarter of the total horizontal distance. Find its tangential acceleration and the radius of curvature of the path at this point.

Problem 22.

Solve Problem 20, assuming that the air exerts a constant resistance in the horizontal direction only which is equal to 0.02 of the weight of the projectile.

Problem 23.

A bullet is fired vertically upward with a velocity of 1000 ft. per sec. from a car moving at a velocity of 40 miles per hour. How far ahead of its starting point will it strike the ground at the same level? If fired at an angle of 60° with the horizontal where will it strike the ground: (a) if the inclination is in the direction of the motion of the car? (b) if the inclination is in the opposite direction?

Problem 24.

A weight of 10 lbs., starting from rest at A, slides down the frictionless plane AO (Fig. 172) and then falls, striking a horizontal plane 15 ft. below O

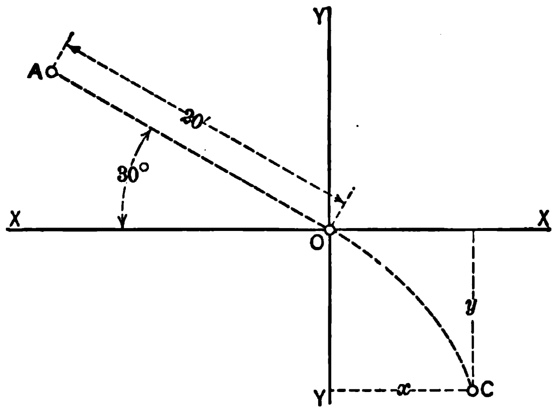

FIG. 172.

at the point C. Find the coördinates of the point C. Find the velocity of the weight at O and also its velocity at the point C. Find the time it will take in going from the point A to the point C.

Problem 25.

Solve Problem 24, assuming that the frictional resistance of the plane AO is a constant and equal to 2 lbs.

Problem 26.

A weight of 10 lbs., attached to a cord 40 ins. long, swings as a pendulum, starting from the position (A) (Fig. 173). Find the tension in the cord: (a) when the weight is in the position (A); (b) when in the position (B); (c) when

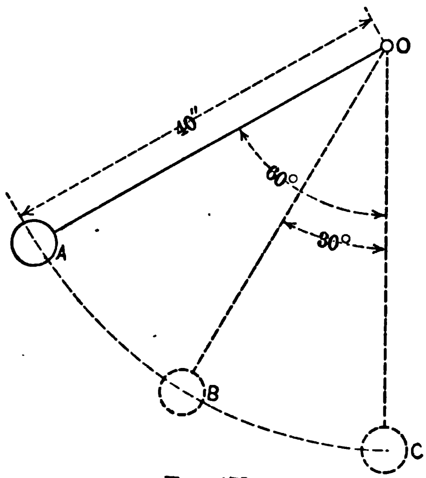

FIG. 173.

in the position (C), on the vertical through the point of suspension. Neglect the weight of the cord. Find the magnitude and direction of the resultant of the forces acting on the weight when in the positions (A), (B) and (C).

Problem 27.

Find the time of oscillation if the pendulum (Prob. 26) starts from rest at the point B : by formula (4) and also by formula (5) (Art. 156).

Problem 28.

A weight w, attached to a flexible string, revolves in a vertical circle of radius r. Neglecting the weight of the string and assuming that gravity is the only force acting, what is the smallest possible velocity of the weight and tension in the string: (a) at the highest point in the path? (b) at the lowest point?

Problem 29.

A weight of 5 lbs. starting from rest at A slides down a slender rod (Fig. 174) in the form of a circular arc of 4 ft. radius. Neglecting friction, find the pressure exerted by the rod on the weight at the points D, B and C, OB being the horizontal through O, the center of the circle. Find the magnitude and direc-

tion of the resultant of the forces acting on the weight when in the positions (*B*) and (*C*).

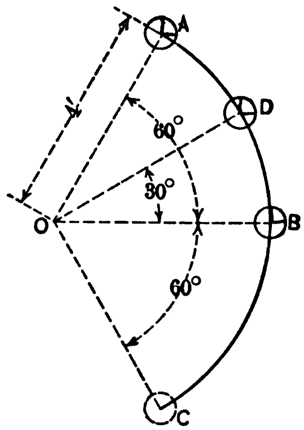

FIG. 174.

Problem 30.

A cylinder 5 ft. in diameter rests on a horizontal plane with its axis parallel to the plane. If a particle of weight w, starting from the highest point on the surface of the cylinder, slides off, without friction, find the point at which it leaves the cylinder; also find the point at which it strikes the plane.

Problem 31.

A body weighing 10 lbs. moves with harmonic motion, of an amplitude of 12 ins. and a period of 0.4 secs. Find the velocity of the weight at its mid-position *O*, and its velocity and the magnitude of the accelerating force at points 4 ins., 8 ins. and 12 ins. from *O*.

Problem 32.

Plot the space-time curve, the velocity-time curve and the acceleration-time curve for a complete cycle of the motion of the weight in Problem 31. With a base line divided to represent displacements plot the acceleration-space and the velocity-space diagrams for the motion.

Problem 33.

The extension of a light helical spring is proportional to the load applied, that is, a weight of 1 lb. will extend it 1 in.; 2 lbs. will extend it 2 ins., etc. If a weight of 5 lbs. is attached and the spring is set to oscillating, by pulling the weight down an additional inch and then releasing it, find the period of the oscillation. Show that the period is independent of the amplitude of the motion.

§ 2. WORK, POWER AND ENERGY.

162. Work. — When a constant force acts in the direction of a body's motion, the work done is the product of the force and the distance, that is

$$W = Fs \text{ (Art. 152)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

While many problems in Engineering may be solved by the use of formula (1), it is frequently necessary to determine the work done by forces which do not act in the direction of the motion and which also may vary either in magnitude, or direction, or both.

Work may be done by a force acting between two bodies which are in contact, as in the case of the pull exerted by a hoisting rope on a weight; or, by forces acting at a distance, as in the case of gravitational, electrical, and magnetic, attractions.

Work may be performed in overcoming frictional resistances and other forces opposing the motion, and also in increasing the velocity and hence the kinetic energy of a body.

Frictional resistances and other forces, opposing the motion of a body, may be treated as forces doing negative work. If a moving body is acted upon by resistances, or retarding forces only, the negative work done will diminish the speed and hence the kinetic energy; or, as we say, the kinetic energy of the body is expended in doing work to overcome the resistances to motion.

When the magnitude of a force is variable and its line of action always coincides with the direction of motion, the work done will be equal to

$$W = \int F\,ds \text{ (Art. 152)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

163. Work Done by an Oblique Force. — If the line of action of a force F (Fig. 175) makes an angle ϕ with OA, the direction of motion of its point of application O at any instant, the force may be resolved into a component $F \cos \phi$, in the direction OA, and a component $F \sin \phi$, perpendicular to OA. Then, according to Art. 162, the work done by the component $F \cos \phi$ during a displacement ds of the point O will be equal to

$$dW = F \cos \phi \, ds; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the work done by the component $F \sin \phi$ will be equal to zero,

since the point O undergoes no displacement in the direction of its line of action.

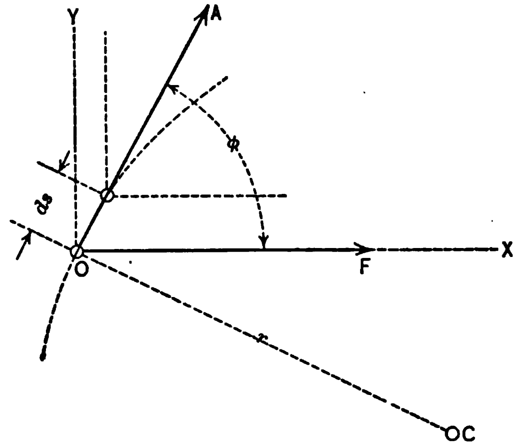

Fig. 175.

Hence the total work performed, during the displacement ds of the point O by the oblique force F, will be equal to the work done by the component $F \cos \phi$; and by adding the successive increments we obtain the expression for the work done during any finite displacement,

$$W = \int F \cos \phi \, ds, \quad \ldots \ldots \quad (2)$$

where F and ϕ may be constant, or variable.

If ϕ is constant,

$$W = \cos \phi \int F \, ds. \quad \ldots \ldots \quad (3)$$

If F is constant,

$$W = F \int \cos \phi \, ds. \quad \ldots \ldots \quad (4)$$

Since the expression for dW (equation 1) may be written

$$dW = F \cos \phi \times ds = F \times \cos \phi \, ds,$$

the work done during the displacement ds is evidently equal to the product of the force F and the projection of ds on its line of action.

It follows from equation (4) that: *When the force F is constant both in direction and magnitude, the work done during any finite displacement will be equal to the product of the force and the projection of the path of its point of application on its line of action.*

164. Work Done by a Rotating Force. — If the path of O, the point of application of F (Fig. 175) is the circumference of a circle, of radius r, we may write

$$ds = r\, d\theta,$$

where $d\theta$ is the angle subtended by the arc ds.

Substituting this value in equation (2) (Art. 163) we obtain

$$W = \int F \cos \phi r\, d\theta = \int M d\theta, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $M = rF \cos \phi$ is the moment of F (Art. 61) about C, the center of rotation at any instant.

When M is a constant the expression for the work done will evidently be

$$W = M\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\theta =$ the total angular displacement of the point of application of the force. For a complete revolution,

$$W = M\, 2\pi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and for N revolutions

$$W = M\, 2\pi N. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It should be noted that the above equations apply when the line of action of the rotating force is not in a plane perpendicular to the axis of rotation, as well as when its line of action is in that plane.

165. Work Done by the Components of a Force. — Let F be any force making the angle ϕ with OA, the direction of the motion of its point of application at any instant (Fig. 176).

Assume any three rectangular coördinate axes OX, OY and OZ, making the angles α, β and γ, respectively, with the line of action of the force F, and resolve F into components $F \cos \alpha = X$, $F \cos \beta = Y$ and $F \cos \gamma = Z$ in the directions of the three axes. Then, during a displacement ds of the point of application of F in the direction OA, the work done by the component in the direction OX will be equal to

$$F \cos \alpha\, dx = X\, dx,$$

where dx is the projection of ds on the line of action of the component $F\ \cos\alpha$ (Art. 163). Similarly the work done by the component in the direction OY will be equal to

$$F\cos\beta\, dy = Y\, dy,$$

and by the component in the direction OZ,

$$F\cos\gamma\, dz = Z\, dz.$$

Adding these three quantities together we obtain

$$X\, dx + Y\, dy + Z\, dz,$$

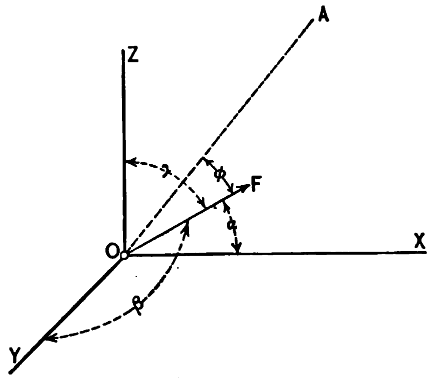

FIG. 176.

which may evidently be written

$$X\frac{dx}{ds}ds + Y\frac{dy}{ds}ds + Z\frac{dz}{ds}ds$$
$$= \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right)ds.$$

But $X\dfrac{dx}{ds}$ is the component in the direction OA, obtained by resolving the force $F\cos\alpha = X$ into a component along OA and one at right angles, and $Y\dfrac{dy}{ds}$ and $Z\dfrac{dz}{ds}$ are the components obtained by resolving $F\cos\beta = Y$ and $F\cos\gamma = Z$ in a similar manner, and hence the sum of these quantities will be equal to the component $F\cos\phi$, obtained by resolving the resultant force F into a component along OA and one at right angles.

Hence

$$dW = F\cos\phi\, ds = \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right)ds$$
$$= X\, dx + Y\, dy + Z\, dz, \quad . \quad . \quad . \quad (1)$$

and, since this relation will hold true for each increment of motion, the work done during any finite displacement will be equal to

$$W = \int F \cos \phi \, ds = \int X \, dx + \int Y \, dy + \int Z \, dz$$
$$= \int F \cos \alpha \, dx + \int F \cos \beta \, dy + \int F \cos \gamma \, dz. \quad . \quad . \quad (2)$$

Therefore the work done by a force during any finite displacement of its point of application is equal to the sum of the works done by its components in any three directions at right angles.

When the resolution is made parallel to two coördinates only, we may substitute $Z = F \cos \gamma = 0$ in equation (2) and obtain

$$W = \int F \cos \phi \, ds = \int X \, dx + \int Y \, dy$$
$$= \int F \cos \alpha \, dx + \int F \sin \alpha \, dy \, . \quad . \quad . \quad . \quad . \quad (3)$$

166. Work Done by a System of Forces. — To determine the work done by any system of forces, we may assume three rectangular coördinate axes OX, OY and OZ and resolve each force into components parallel to the three axes. The expression for the work done by any force in the system, during any finite displacement of its point of application, may then be written in the form of equation (2) (Art. 165) and the summation of the works done by all the forces in the system can be represented by the expression

$$\Sigma W = \Sigma \int F \cos \phi \, ds = \Sigma \left[\int X \, dx + \int Y \, dy + \int Z \, dz \right]$$
$$= \Sigma \int X \, dx + \Sigma \int Y \, dy + \Sigma \int Z \, dz. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When the forces act at a single point, or at different points which move in the same direction with the same velocity at every instant, as when the forces act on a rigid body having a motion of translation (Art. 201), the displacement ds and the components dx, dy and dz, during any increment of time dt, will be the same for every force in the system and hence equation (1) may be written

$$\Sigma W = \int \left(\Sigma F \cos \phi \right) ds = \int \left(\Sigma X \right) dx + \int \left(\Sigma Y \right) dy + \int \left(\Sigma Z \right) dz$$
$$= \int R \cos \phi_r \, ds, = \int X_r \, dx + \int Y_r \, dy + \int Z_r \, dz, \quad . \quad . \quad (2)$$

where

$$R \cos \phi_r = \Sigma F \cos \phi,\ X_r = \Sigma X,\ Y_r = \Sigma Y, \text{ and } Z_r = \Sigma Z$$

represent the vector sums of the components of the forces at any instant in the direction of the motion and of the coördinate axes, X, Y, Z respectively.

When the forces in a system act at different points which rotate about the same axis with the same angular velocity at each instant, as in the case of a system acting at different points on a rigid body having a motion of rotation (Art. 205), we may deduce the expression for the work done by the resultant of the system as follows:

Let F, F_1 and F_2 be any system of forces, not necessarily in the same plane, acting at the points O, O_1 and O_2, which at a given instant rotate with the same angular velocity about an axis through A (Fig. 177). Let R be the resultant of the system and O_r, its point of application; and let M, M_1, M_2 and M_r be the moments (Art. 61) of the forces F, F_1, F_2 and R, respectively, about the axis A. Then for an angular displacement $d\theta$ of the points O, O_1, O_2 and O_r, the works done by the forces F, F_1, F_2 and R will be equal to $M\,d\theta$, $M_1\,d\theta$, $M_2\,d\theta$ and $M_r\,d\theta$, respectively (Art. 164).

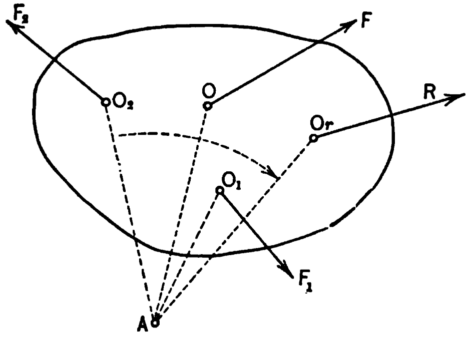

Fig. 177.

Since the sum of the moments of the forces about any axis must equal the moment of their resultant about that axis (Art. 71), we have

$$M_r = M + M_1 + M_2.$$

Multiplying by $d\theta$,

$$M_r\,d\theta = M\,d\theta + M_1\,d\theta + M_2\,d\theta.$$

Hence for each increment of the motion, the work done by the resultant is equal to the algebraic sum of the works done by the forces in the system and, by adding the successive increments of work during any finite displacement, we obtain

$$\Sigma W = \int M_r\, d\theta = \int M\, d\theta + \int M_1\, d\theta + \int M_2\, d\theta = \Sigma \int M\, d\theta, \qquad (3)$$

in which equation M, M_1, M_2 and M_r may be constant, or variable.

It may be noted, if the direction of the motion in this case is that indicated in Fig. 177, the force F_2 will act as a resistance to the motion and M_2 will be a negative quantity.

Hence in each of the above cases, *the work done by the resultant of a system of forces is equal to the algebraic sum of the works done by the forces in the system, the work done by any force tending to retard the motion being considered negative.*

When the system comprises all the forces acting on the body, if we let W_a = the total positive work done by the forces tending to accelerate the motion and W_r = the total negative work done by the forces tending to retard the motion, during a given displacement, it is evident from the above that the work done by the resultant of the entire system may be represented by the expression

$$W = W_a - W_r. \qquad (4)$$

It will be shown later that in this case W is equal to the change produced in the kinetic energy of the body by the system of forces acting upon it.

Hence, by transposing, we have

$$W_a = W + W_r. \qquad (5)$$

that is, the total work done by the forces tending to accelerate the motion of the body during a given displacement will be equal to the change produced in the kinetic energy plus the work done in overcoming the forces resisting the motion.

167. Work Done by a Rotating Couple. — This is a special case under Art. 166 (equation 3) where the resultant of the system of forces is a couple, instead of a single force, and hence the equation for the work done will be

$$W = \int M_r\, d\theta, \qquad (1)$$

where M_r, the moment of the resultant couple at any instant, will be equal to the algebraic sum of the moments of the forces about the axis of rotation.

In the case of a single couple, in a plane perpendicular to the axis of rotation, the equation for the work done during any displacement may be deduced independently, as follows: Let $F_1 = F_2$ be the forces of the couple and let M_1 and M_2 be their respective moments at any instant about O, the axis of rotation (Fig. 178). Let $M = M_1 + M_2$ equal the moment of the couple.

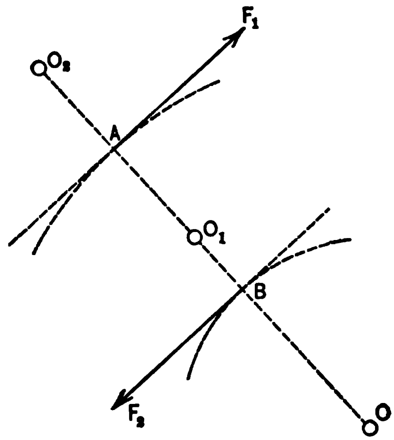

FIG. 178.

Multiplying by $d\theta$, we obtain

$$M\,d\theta = M_1\,d\theta + M_2\,d\theta.$$

Hence, during each increment of motion, the work done by the couple will be equal to the algebraic sum of the works done by the forces F_1 and F_2 and, adding the successive increments of work, we obtain

$$W = \int M\,d\theta = \int M_1\,d\theta + \int M_2\,d\theta. \quad . \quad . \quad . \quad . \quad (2)$$

The above will evidently apply equally well if the axis of rotation is at O_1 or O_2, the signs of M_1 and M_2 depending on the position of the axis relatively to the forces F_1 and F_2.

When M is a constant, the work done by the couple during an angular displacement θ will be equal to

$$W = M\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the work done during N revolutions will be equal to

$$W = M\,2\,\pi N. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

When rotation is produced by a couple, in the manner indicated above, the moment of the couple is called the *turning moment,* or *torque.*

168. Units of Work. — The unit of work is the work which is done when a unit force acts through a unit distance in the same direction as the force. In the British system, when the units of force and distance are the pound and foot respectively, the unit of work is called the *foot-pound.*

If large quantities are dealt with, the units of force and distance are sometimes taken as the ton and foot, in which case the unit of work is called the *foot-ton.*

In the French system, when the units of force and distance are the kilogram and meter respectively, the unit of work is called the *meter-kilogram.* One meter-kilogram = 7.233 foot-pounds.

In the c.g.s. system, the unit of work usually employed is the *erg,* which may be defined as the work done by a force of 1 dyne acting through a distance of 1 centimeter.

169. Work Diagram. — When the line of action of a force coincides with the direction of the motion of its point of application at every instant, if a diagram is made by plotting the magnitude of the force, at different points in its path, as ordinates and the

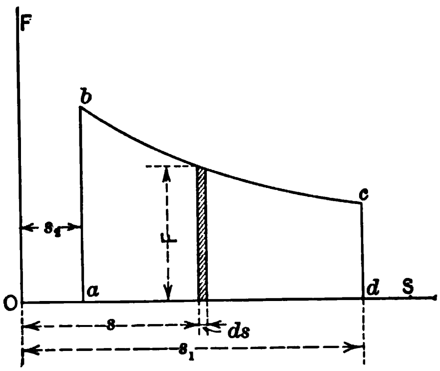

FIG. 179.

corresponding distances passed over as abscissæ (Fig. 179), the area (*abcd*), under the line drawn through the extremities of the ordinates, will represent the work done. The area under *bc* will evidently be equal to $W = \int_{s_2}^{s_1} F\, ds$, where s_1 and s_2 are the dis-

tances from the origin to the points between which the work is to be computed.

If the direction of the force does not always coincide with the direction of motion, the diagram may be made by plotting the value of $F \cos \phi$, the component of the force in the direction of the motion at different points in the path, as ordinates and the corresponding distances passed over as abscissæ. The area under the line, drawn through the extremities of the ordinates, will then be equal to

$$W = \int_{s_2}^{s_1} F \cos \phi \, ds \text{ (Art. 163)},$$

where s_1 and s_2 are the distances from the starting point, measured along the path, and ϕ is the angle between the line of action of the force and the tangent to the path at any instant.

In the case of rotary motion, if a diagram is made in which the ordinates are values of M, the moment of the force about the axis of rotation at an instant, and the abscissæ are the corresponding values of θ, the angular displacement of its point of application, the area under a line through the ends of the ordinates will be equal to

$$W = \int_{\theta_2}^{\theta_1} M d\theta \text{ (Art. 164)},$$

which will be the work done during an angular displacement $(\theta_1 - \theta_2)$.

170. Power. — Power is the rate of doing work. By the power of a machine is meant the amount of work which it is capable of doing in a given time. If the rate is constant the power may be expressed by the formula

$$P = \frac{W}{t}, \quad \ldots \ldots \ldots \quad (1)$$

where $W =$ the work done in the time t; and if the rate is varying, by the expression

$$P = \frac{dW}{dt}. \quad \ldots \ldots \ldots \quad (2)$$

If the work is being done by a force F, acting in the direction of the motion, equation (2) may be written

$$P = \frac{Fds}{dt} = Fv, \quad \ldots \ldots \ldots \quad (3)$$

that is, the power exerted at any instant will be equal to the product of the force and the velocity.

In the case of a rotating force, or couple, whose moment at any instant about the axis of rotation is equal to M, equation (2) may be written

$$P = \frac{dW}{dt} = \frac{M\,d\theta}{dt} = M\omega, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

that is, the power exerted at any instant will be equal to the product of the moment of the force, or couple, and the angular velocity.

171. Units of Power. — In the British system of units, the unit of power is the *foot-pound per second.*

Where large amounts of power are measured the unit most commonly employed is the *horse-power* which is equal to 33,000 foot-pounds per minute, or 550 foot-pounds per second.

If the force doing the work acts in the direction of the motion, we may obtain from equation (3) (Art. 170) the following expression for the horse-power exerted,

$$\text{h.p.} = \frac{Fv}{550},$$

which will evidently hold true for either a constant or varying force. If the work is being done by a rotating force, or a couple, we may obtain from equation (4) (Art. 170) the expression for horse-power,

$$\text{h.p.} = \frac{M\omega}{550} = \frac{M\,2\pi N}{33{,}000},$$

where N = the number of revolutions per minute.

In the French system of units, the unit of power is the *meter-kilogram per second.* The horse-power (*force de cheval*) in this system is a slightly different quantity from the horse-power in the British system, being equal to 75 meter-kilograms per second. This unit is equivalent to 32,550 foot-pounds per minute, or 0.986 h.p. in British units.

In electrical engineering the units of power commonly used are the *watt* and the *kilowatt*, the equivalents being

$$1 \text{ watt} = 10^7 \text{ ergs per sec.}$$

$$1 \text{ kw.} = 1000 \text{ watts} = 1.340 \text{ h.p.},$$

and conversely,

$$1 \text{ h.p.} = 0.746 \text{ kw.}$$

Derived Units. From these units of power other units for measuring large quantities of work are derived, such as the horse-power hour (h.p. hr.), the kilowatt hour (kw. hr.), etc. As the name indicates, the horse-power hour is one horse-power exerted for one hour, hence

$$1 \text{ h.p. hr.} = 33{,}000 \times 60 = 1{,}980{,}000 \text{ ft.-lbs.}$$

Similarly,

$$1 \text{ kw. hr.} = 1.340 \times 1{,}980{,}000 = 2{,}650{,}000 \text{ ft.-lbs.}$$

172. Power Diagram. — A power diagram may be made in the same manner as a work diagram (Art. 169) by plotting units of power as ordinates and distances as abscissæ. Such a diagram will show the variation in power or, in the rate at which work is being done by a force, at different points in the path of its point of application.

A power diagram may also be made by plotting units of power as ordinates and times as abscissæ. Such a diagram will show the variation in power during different intervals of time; and the area under the curve between any two ordinates will represent the work done during the interval of time measured between the ordinates. For example, if British units are used, the area will be equal to

$$\int (\text{h.p.})\, dt = \int \frac{Fv}{550} dt = \frac{1}{550} \int F\, ds,$$

which will be $\frac{1}{550}$ of the work in foot-pounds, performed during the interval of time.

173. Kinetic Energy of a Particle. — The expression for the change in the kinetic energy of a freely moving particle, produced by a force acting in the direction of its motion, is

$$\int F\, ds = \frac{m}{2} (v_1^2 - v_0^2) \text{ (Art. 152)}. \quad . \quad . \quad . \quad . \quad (1)$$

When the line of action of the force does not coincide with the direction of motion of the particle, if the force is resolved into a tangential component $F_t = F \cos \phi$ and a normal component $F_n = F \sin \phi$, it is evident that the component F_n will produce no effect on the speed: and hence, from the equation

$$F_t = F \cos \phi = m \frac{dv}{dt} \text{ (Art. 150)},$$

we may deduce the expression

$$\int F \cos \phi \, ds = \frac{m}{2}(v_1^2 - v_0^2) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

by the same method as was followed in the deduction of equation (1).

Hence the change in kinetic energy during a given displacement of the particle under the action of an oblique force will be equal to the work done by the force.

If the particle is acted upon by a system of forces, it is evident that, if each force is resolved into tangential and normal components, the vector sum of the tangential components will be equal to

$$\Sigma F \cos \phi = m \frac{dv}{dt},$$

and the normal components will have no effect on the speed of the particle. Hence, if R = the resultant of the system and ϕ_r = the angle between its line of action and the direction of motion at any instant, we shall have

$$R \cos \phi_r = \Sigma F \cos \phi = m \frac{dv}{dt}$$

and hence,

$$\int R \cos \phi_r \, ds = \frac{m}{2} (v_1^2 - v_0^2). \quad . \quad . \quad . \quad . \quad (3)$$

Therefore, the change in the kinetic energy of a particle, during a given displacement under the action of a system of forces, will be equal to the work done by the resultant of the system.

If $v_0 = 0$, and we let $v_1 = v$,

$$\int R \cos \phi_r \, ds = \frac{mv^2}{2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

that is, the kinetic energy of a particle is equal to the work done by the resultant of the system of forces acting on the particle during the change in velocity from 0 to v.

Since rest and motion are relative quantities we have no means of determining the absolute kinetic energy of a particle but only

the energy due to its motion with reference to some fixed point.

174. Kinetic Energy of a System of Particles.—If we assume the system of particles to be acted upon by a system of forces, the change in the kinetic energy of any particle during an interval of time, $t - t_0$, may be represented by the expression

$$\int R \cos \phi_r \, ds = \frac{mv^2}{2} - \frac{mv_0^2}{2} \text{ (Art. 173)}, \quad . \quad . \quad . \quad (1)$$

where R = the resultant of the forces acting on the particle, ϕ_r = the angle between its line of action and the direction of motion at any instant, v_0 = the velocity at the time t_0, and v = the velocity at the time t. Since equation (1) will represent the change in energy of any particle in the system, the total change in the kinetic energy of the system may be represented by the summation

$$\Sigma \int R \cos \phi_r \, ds = \Sigma \left[\frac{mv^2}{2} - \frac{mv_0^2}{2}\right] = \Sigma \frac{mv^2}{2} - \Sigma \frac{mv_0^2}{2}, \quad . \quad (2)$$

where the first member of the equation represents the total work, ΣW, performed by the system of forces (Art. 166) during the displacement of the particles along their different paths in the time $t - t_0$, and the second member represents the difference in the sum of the energies of the particles in the system at the time t_0, and the sum of the energies at the time t. Hence, if we represent these quantities by E_0 and E, respectively, we shall have

$$\Sigma W = E - E_0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

that is, the change in the total kinetic energy of a system of particles during a given interval of time is equal to the work done by the system of forces acting upon the particles during that interval of time.

When all the particles move with the same velocity at every instant, it is evident that equation (2) may be written

$$\Sigma W = \frac{v^2 - v_0^2}{2} \Sigma M. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the resultant force R, acting on any particle in the system, is resolved into components, X_r, Y_r and Z_r, parallel to three rectangular coördinate axes, OX, OY and OZ; then, by treating each

component of R separately, we may obtain by the method employed in deducing equation (2) (Art. 173)

$$\int X_r\,dx = \frac{m}{2}(v_x^2 - v_{0_x}^2),$$

$$\int Y_r\,dy = \frac{m}{2}(v_y^2 - v_{0_y}^2),$$

$$\int Z_r\,dz = \frac{m}{2}(v_z^2 - v_{0_z}^2),$$

where v_{0_x}, v_{0_y}, v_{0_z} and v_x, v_y, v_z are the components, in the directions OX, OY and OZ, of v_0 and v, the initial and final velocities of the particle.

Making the summation for all the particles in the system we have

$$\Sigma \int X_r\,dx = \Sigma \frac{m}{2}(v_x^2 - v_{0_x}^2), \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\Sigma \int Y_r\,dy = \Sigma \frac{m}{2}(v_y^2 - v_{0_y}^2), \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\Sigma \int Z_r\,dz = \Sigma \frac{m}{2}(v_z^2 - v_{0_z}^2), \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where equations (5), (6) and (7) represent the change in energy of the system produced by the component forces, parallel to the axes OX, OY and OZ, respectively.

By adding the three equations together and noting that for each particle

$$v_x^2 + v_y^2 + v_z^2 = v^2 \quad \text{and} \quad v_{0_x}^2 + v_{0_y}^2 + v_{0_z}^2 = v_0^2,$$

we may easily deduce equation (2) representing the total change in energy of the system. Since the kinetic energy of a particle or system of particles is the power of doing work which is due to velocity, the *units of energy* are the same as the units of work (Art. 168).

175. Potential Energy. — As stated in Art. 152, mechanical energy is ordinarily considered to be of two kinds: kinetic energy, or energy of motion, and potential energy, or energy of position.

The latter may be considered to be divided into two classes: first, the energy due to the position of two or more bodies, between which there are exerted attracting or repelling forces, which is commonly called the energy of position, or the energy of configuration; and second, the energy due to a distorted or strained condition of a body, which may be called the energy due to strain, or strain energy.

A familiar example of the first of these two classes of potential energy is that which is due to the position of a weight suspended above the Earth. Work will be done by the attraction of the force of gravity as the weight is allowed to approach the Earth, and the amount of work done as the weight moves from one level to another will be equal to the difference in potential energy between the two levels.

In this case, it is convenient to consider the potential energy as being possessed by the weight: that is, the weight, when at a given height above the Earth's surface, is said to possess a certain potential energy above that which it would have at the surface. Thus, when a weight W is raised to a height h, above some level which may be taken as a standard, we say that the *potential energy of the weight* is equal to Wh. To be exact, however, we must consider the energy as being possessed by the weight and the Earth together, since it is the power of doing work which is due to the existence of the mutual attraction of gravity between the two, and is not due to any property, or condition, possessed by either one alone.

The above is an example of potential energy due to the relative position of *two* bodies between which a force of attraction exists. Another example of this kind would be that of an iron weight under the attraction of a magnet. If the magnet is considered to be fixed, we may say that the *potential energy of the weight*, in any position, is equal to the work done by the force of magnetic attraction as the weight moves to a position in contact with the magnet.

In a similar manner two particles under the action of a mutual attracting force, such as gravitation, will possess an amount of potential energy dependent on the distance between them. For convenience we may assume one of the particles to be fixed and assign the potential energy of the system to the other particle.

Another illustration of the first form of potential energy is the case of a system of more than two particles which move under the action of forces of attraction which exist between them. In this case the potential energy of the system will be dependent on the relative positions of the particles, or the configuration of the system, and hence is called the *energy of configuration*. If a given configuration is taken as a standard, the potential energy of the system for any other configuration will be equal to the work which can be done as it passes from that configuration to the standard

one. Hence, the energy may be represented by the general expression for the work done by the forces acting between the particles, during the change in configuration, viz.:

$$\Sigma W = \Sigma \int F \cos \phi \, ds \text{ (Art. 166).}$$

In this case it would not be possible, in general, to assign any definite value to the energy of any one particle, or group of particles. Exceptions in the case of two particles, or groups of particles comprising two rigid bodies, have been noted above.

An example of the second class of potential energy is that of a spring which is distorted from its normal position or shape, by the action of external forces. When the forces are removed, the spring will return to its original shape, doing a certain amount of work. The potential energy of the distorted spring may be called *strain energy*. If we conceive of the spring as being composed of small particles which are displaced from their normal, or standard of positions, and between which internal forces due to strains are exerted when the spring is distorted, the energy may be considered to be one of configuration, similar to the preceding case.

An important difference, however, between this case and the previous ones is that here the potential energy may be definitely located in the spring. Each portion of the spring, when distorted, will perform an amount of work in returning to its normal condition, which may be definitely determined, and the total potential energy of the spring will be equal to the sum of the energies of its parts.

176. Conservative System. — If a system of particles is moved from one position to another by the action of a system of forces, the work done by the system can be expressed in terms of the forces acting and the distances moved through by the particles, Art. 166. In general, the work done will depend on the paths of the particles from the initial to the final positions.

There is a class of cases, however, in which the work done by the forces, as the particles pass from one set of positions to another, is independent of the paths of the particles and depends only on the initial and final positions. In such a case, the system of forces acting is said to be a *conservative system* and the forces may be called *conservative forces*.

An example of a conservative system is that of a system of particles moving under the action of the forces of attraction or

repulsion existing between them (Art. 85). The force F, exerted between any two particles of such system, is a function of the distance between the particles only, and, for a relative displacement ds, of one of the particles with respect to the other, the work done by F will be equal to

$$F \cos\phi \, ds \text{ (Art. 164)},$$

where $\cos \phi \, ds$ is equal to the change in the distance between the two particles during the displacement ds. Hence the work done by F, during the increment of motion, *is a function of the masses and the distance between the particles only;* and, as this condition will hold for every increment of motion and for all the forces exerted between the particles, the total work done during any finite displacement of the system will depend only on the initial and final positions of the particles.

Therefore, if we let R = the resultant of the forces exerted on any particle in the system at any instant, and ϕ_r, the angle between R and the direction of motion of the particle, the change in potential energy between one configuration of the system and another (Art. 175) may be represented by the expression

$$\Sigma W = \Sigma \int F \cos\phi \, ds = \Sigma \int R \cos\phi_r \, ds.$$

This is evidently equal to the change in kinetic energy of the system during the displacement of the particles from the one set of positions to the other (Art. 174). Hence, if we let E_{p_0} and E_{k_0} equal the potential and kinetic energies of the system in any standard configuration, and E_p and E_k, the potential and kinetic energies in any other configuration, we shall have

$$E_p - E_{p_0} = E_{k_0} - E_k$$

or,

$$E_{p_0} + E_{k_0} = E_p + E_k; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

that is, *the sum of the potential and kinetic energies of a system of particles moving under the action of conservative forces is the same for all configurations of the system.*

When the forces acting on the particles in a system are such that the quantity

$$\Sigma \int R \cos\phi_r \, ds$$

depends on the paths through which the particles move in passing

from one configuration to another equation (1) will not hold. In such a case the system is called *non-conservative* and the forces are non-conservative forces. Forces of friction are examples of such forces.

The work done in moving a system of particles from one position to another under the action of such a system of forces would evidently depend on the paths of the particles, since the work done by the force exerted by friction on any particle will be a function of the distance through which the particle moves in going from one position to another.

A special case, which may be cited as an example of a conservative system, is that of the gravitational attraction at a point O (Art. 85), by the particles in a given mass M upon a particle of unit mass m at O. The forces of attraction are conservative forces and the field in which they are exerted may be called a *conservative field.* In this case the mass M may be considered as fixed and the potential energy of the system for any position of the particle m may be assigned to the particle. If the particle moves in the field the sum of its kinetic and potential energies will be a constant.

A simple illustration of the above would be that of the projectile (Art. 158), moving in the field of the Earth's attraction, provided the resistance of the air could be removed. If we let $E_p = wh$ equal the potential energy of the projectile at the height h, the potential energy at the height h_0 will be equal to $E_{p_0} = wh_0$, and the change in potential energy between the two levels will be equal to

$$E_p - E_{p_0} = w(h - h_0).$$

If we resolve the velocity of the projectile, when it has reached the height h, into horizontal and vertical components, v_h and v_v, its kinetic energy will be equal to

$$E_k = \frac{m}{2}(v_h^2 + v_v^2) \text{ (Art. 174).}$$

Similarly, its kinetic energy at the height h_0 will be equal to

$$E_{k_0} = \frac{m}{2}(v_h^2 + v_{0v}^2),$$

the horizontal component of velocity remaining unchanged and the vertical component becoming v_{0v}.

But

$$v_v = \sqrt{2g(x-h)}$$

and

$$v_{0_v} = \sqrt{2g(x-h_0)},$$

where x equals the height at which the vertical component of the velocity is zero. Hence,

$$E_k - E_{k_0} = \frac{w}{2g}(v_v^2 - v_{0_v}^2) = wh_0 - wh = E_{p_0} - E_p. \quad . \quad (2)$$

If the resistance of the air is taken into account the system of forces acting on the particle becomes non-conservative and it is evident that equation (2) will not hold true.

177. The Potential. — Let m_1, m_2, m_3, etc., be the masses of any system of attracting particles which are situated at distances r_1, r_2, r_3, etc., from a given point. The potential of the system at the point may be defined as

$$V = \frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_3}{r_3} + \cdots = \Sigma \frac{m}{r}, \quad . \quad . \quad . \quad . \quad (1)$$

where the distances r_1, r_2 and r_3, are all taken as positive quantities.

We will consider very briefly the meaning of this function in the case where the attracting forces follow the law of gravitation

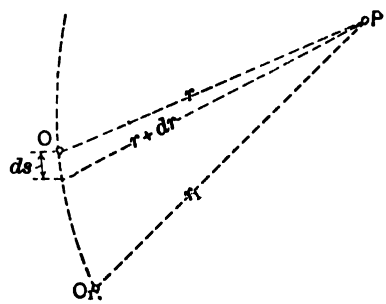

FIG. 180.

(Art. 85). Let m and m_1 be the masses of two particles, at the points P and O respectively (Fig. 180), and r equal the distance between them. Then, for any displacement ds of m_1 with respect

to m, r will become $r + dr$ and the work done by the attracting force during the displacement will be equal to

$$dW = F \cos \phi \, ds = - K \frac{mm_1}{r^2} dr \text{ (Arts. 163 and 85)}, \quad . \quad (2)$$

where $dr = \cos \phi \, ds$, the projection of the displacement on the line of action of the force, and $F = K \frac{mm_1}{r^2}$, the magnitude of the force acting between the particles.

Hence the work done, during the displacement of m_1 along any path to the point O_1, at a distance r_1 from P, will be equal to

$$W = \int F \cos \phi \, ds = - \int_r^{r_1} K \frac{mm_1}{r^2} dr = Kmm_1 \left(\frac{1}{r_1} - \frac{1}{r} \right), \quad (3)$$

which is evidently independent of the path of m_1 between its initial and final positions. If we let m_1 = unity and choose the unit of mass, so that $K = 1$, equation (3) becomes

$$W = \int F \cos \phi \, ds = m \left(\frac{1}{r_1} - \frac{1}{r} \right). \quad . \quad . \quad . \quad . \quad (4)$$

If the point O_1 is removed to an infinite distance, the work done will be equal to

$$W = - \frac{m}{r} .$$

The quantity $\frac{m}{r}$ is the potential of the mass m at the point O, as defined above. Hence, the potential of a particle of mass m at a point O is equal to the work which would be done by its attraction on a particle of unit mass, if the particle were brought up to O along any path from infinity.

If the unit mass were moved from an infinite distance up to the point O under the attractions of a system of particles m_1, m_2, m_3, etc., whose distances from O were the constant quantities r_1, r_2, r_3, etc., the work done would evidently be equal to

$$\frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_3}{r_3} = \Sigma \frac{m}{r} = V. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence, *the potential of a system of attracting particles at a given point O may be defined as the work which would be done by the attractions exerted by the particles in the system in bringing a particle of unit mass up to the point O from infinity; the system of par-*

ticles being a finite one and its configuration remaining constant during the change. When the system of particles forms a continuous mass, the expression for the potential at O may be written

$$V = \int \frac{dM}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which is the general expression for the potential at any point in the field of gravitational attraction surrounding any solid body.

Returning to equation (4): the work done, as the unit mass moves along any path between points whose distances from the mass m are equal to r and r_1, is evidently equal to the difference of the potentials of the mass m at the two points. As this holds true for every particle in a system, it will follow that the work done by the attraction of a system of particles on a particle of unit mass, as it moves from one point to another, will be equal to the difference of the potentials at the two points; that is,

$$W = \Sigma \int F \cos \phi \, ds = \int R \cos \phi_r \, ds = V_1 - V, \quad . \quad (7)$$

where R = resultant attraction of the particles in the system on the unit mass, and ϕ_r = the angle between the tangent to its path and the line of action of R at any point.

Differentiating, we obtain

$$dW = R \cos \phi_r \, ds = dV. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Hence

$$\frac{dV}{ds} = R \cos \phi_r; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

that is, the derivative of the potential at a point with respect to any displacement is equal to the component of the attraction in the direction of the displacement.

In a similar manner, if we resolve R into components X_r, Y_r, Z_r, along three coördinate axes OX, OY and OZ, and take the displacement of the unit mass along each of the axes in turn we shall have

$$\frac{dV}{dx} = X_r, \quad \frac{dV}{dy} = Y_r, \quad \frac{dV}{dz} = Z_r. \quad . \quad . \quad . \quad (10)$$

An imaginary surface which is the locus of all the points in a field of force, at which the potential has a given value, is called a *level surface*, or an *equipotential surface*. Any line in such a surface is called an *equipotential line*.

It is evident that for every point in such a surface

$$\frac{dV}{ds} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and that the work done as a particle moves from point to point in the surface will be equal to zero. It follows from equation (9) that such a surface will be everywhere perpendicular to the *lines of force* (Art. 86) and, if the surface were impenetrable, a particle placed upon it would be in equilibrium under the normal pressure of the surface and the attractive forces.

The potential at any point in the field of attraction of a homogeneous solid may be determined by the integration of equation (6). In the case of the solid, or hollow, sphere; since the attraction at any point outside the surface is the same as if the mass of the sphere were concentrated at its center (Art. 86), it follows from the definition that the potential at any point, outside the surface, at a distance r from the center, will be equal to

$$V = \frac{M}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where M = the mass of the sphere.

For a thin hollow sphere, of radius a and mass M, since the resultant attraction R at all points inside the surface is equal to zero (Art. 86), it follows from equation (9) that the potential at any point inside the surface will be equal to the potential at a point on the surface, and hence

$$V = \frac{M}{a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

It is evident that all equipotential surfaces outside the surface of a sphere will be concentric spheres.

178. Problems.—Work, Energy, Power.—In any of the following problems, involving mass, each body may be treated as if it were a single particle. It will be shown later that no error is introduced by making this assumption.

Problem 1.

Find the work done by the force F in moving the weights in Problem 3 (Art. 161) through a distance of 12 ft. If the weights start from rest, find the kinetic energy of each weight after moving that distance.

Problem 2.

Find the kinetic energy of the weights in Problem 7 (Art. 161) after moving a distance of 5 ft. from a position of rest.

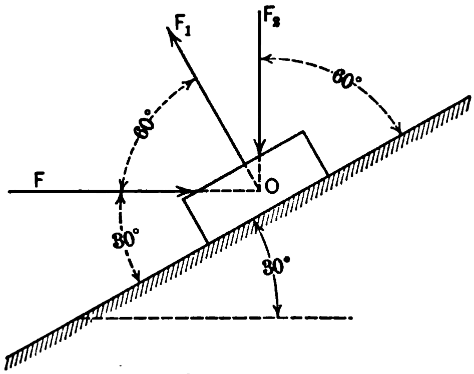

FIG. 181.

Problem 3.

A weight of 40 lbs., starting from rest, is moved along a frictionless inclined plane (Fig. 181) under the action of the forces F, F_1, F_2, the reaction of the plane and gravity. Assuming that the forces remain constant in magnitude and direction, and that $F = 60$ lbs., $F_1 = 30$ lbs. and $F_2 = 20$ lbs., find the kinetic energy of the weight at the end of 20 secs. Find the work done and the maximum power exerted during the 20 secs. by the force F; by the force F_1; by the force F_2.

Problem 4.

Solve Problem 3, assuming that the force $F_2 = 80$ lbs., the other forces remaining the same.

Problem 5.

Two weights, of 40 lbs. each, rest on inclined planes, making angles of 30 and 60° with the horizontal, and are connected with a flexible cord running over a pulley at C (Fig. 182). If the weights start from rest, find the kinetic energy of each one after moving through a distance of 10 ft., neglecting friction and the weight of the pulley and cord.

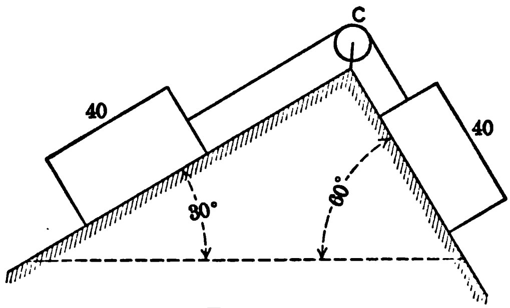

FIG. 182.

Problem 6.

Assuming a force of friction equal to 4 lbs. between each weight and the plane on which it rests (Fig. 182), find the kinetic energy of the system after 5 secs. if the initial velocity of the weights is equal to 10 ft. per sec.

Problem 7.

A weight of 15 lbs., starting from rest, is raised vertically by a force whose initial magnitude is 25 lbs. and which decreases uniformly at the rate of 1 lb. per ft. of distance through which the weight is raised. Find the height to which the weight will rise under the action of such a force.

Problem 8.

The work diagram for the resultant of the forces moving a weight of 20 lbs. along a horizontal plane is shown in Fig. 183. If the velocity of the weight is zero at A, find its velocity at B. Find the power exerted when the weight is at C; when it is at B.

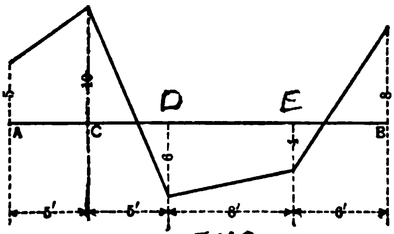

FIG. 183

Problem 9.

Solve Problem 8, assuming that the velocity at A is 20 ft. per second.

Problem 10.

Find the work done in 10 secs. by the force F in Problem 9 (Art. 161).

Problem 11.

Find the work done in 10 secs. by the force F in Problem 10 (Art. 161).

Problem 12.

Plot the work diagram for the force F in Problem 10 (Art. 161) for the space traversed in 10 secs.

Problem 13.

Plot the work diagram for the force acting in Problem 14 (Art. 161) for a distance of 500 ft. Find the kinetic energy of the weight after moving 500 ft.

Problem 14.

Assuming that the moving weight in Problem 12 (Art. 161) is 10 lbs., find the work done by the resultant of the forces acting upon it during the first 12 secs. Plot the work diagram for the resultant of the forces acting during the first 20 secs.

Solve graphically, measuring distances at 1 sec. intervals.

Find the maximum power exerted and the maximum kinetic energy of the weight during that time.

Problem 15.

Find the kinetic energy of the projectile in Problem 20 (Art. 161) at the starting point; when it is at its highest point; when it strikes the ground.

Problem 16.

Find the kinetic energy of the weight in Problem 26 (Art. 161) when it reaches its lowest position at C.

Problem 17.

Find the work done by the force acting on the weight in Problem 31 (Art. 161) as it moves through a distance of 1 ft. from its mid-position.

Problem 18.

A pulley, 4 ft. in diameter, rotates under the action of the forces shown in Fig. 184. Find the work done by each of the forces during 100 revolutions of the pulley; also by the resultant of the system.

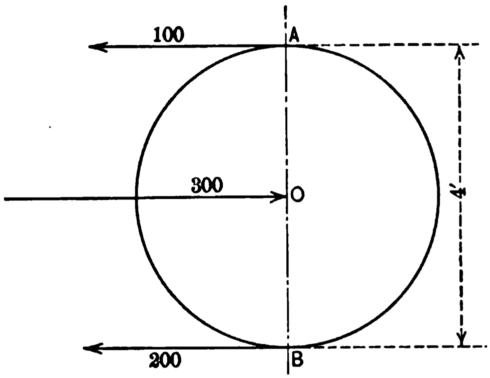

Fig. 184.

Problem 19.

A wheel 2 ft. in diameter is acted upon at A and B by two constant, equal and opposite forces of 10 lbs. each (Fig. 185), the lines of action, of

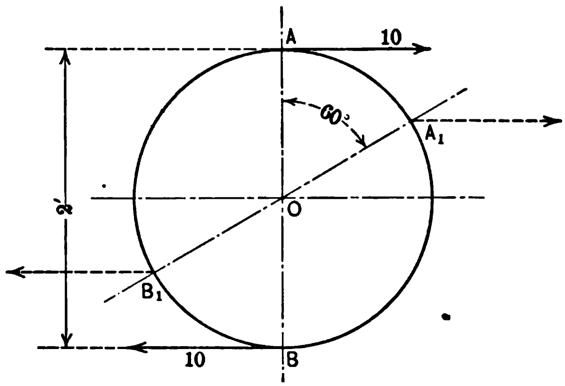

Fig. 185.

which, remain horizontal as the wheel turns. Find the work done in turning the wheel through an angle of 60°, as indicated in the figure.

Problem 20.

A wheel is acted upon by two constant, equal and opposite horizontal forces of 20 lbs. each (Fig. 186), the lines of action, of which, remain horizontal as the wheel turns. Find the work done in turning the wheel through an angle of 45°, as indicated in the figure.

Problem 21.

Solve Problem 19 assuming that the forces remain tangent to the circumference as the wheel turns. Find the work per revolution in this case.

Problem 22.

Solve Problem 20 assuming that the forces remain perpendicular to the radius OA as the wheel turns. Find the work per revolution in this case.

Problem 23.

Plot the power diagram for Problem 8, using distances as abscissæ; using times as abscissæ.

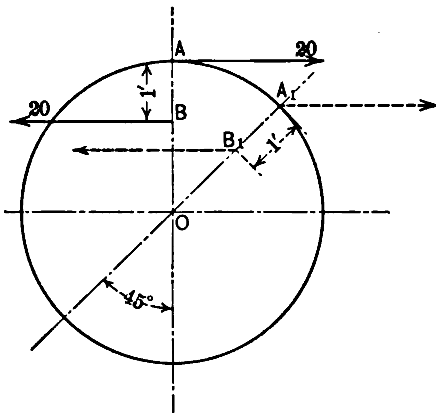

Fig. 186.

Problem 24.

Plot the power diagram for Problem 13, using distances as abscissæ; using times as abscissæ.

Problem 25.

A motor-driven shaft, turning at 500 revolutions per minute, is subjected to a constant torque of 500 ft.-lbs. Find the horse-power transmitted.

Problem 26.

A shaft carries 25 horse-power, turning at a speed of 300 revolutions per minute; find the torque.

Problem 27.

If the allowable torque on a shaft is 200 ft.-lbs. find the speed in revolutions per minute necessary for the shaft to transmit 20 horse-power

Problem 28.

A locomotive exerts a pull of 16,000 lbs. at the draw-bar when traveling at a speed of 30 miles per hour. Find the horse-power exerted.

Problem 29.

A weight of 1000 lbs. is raised at a uniform speed of 200 ft. per minute by a hoisting engine. Find the horse-power exerted.

§ 3. Friction.

179. Sliding Friction. — When one body slides, or tends to slide, over the surface of another, the resistance to the motion which is developed is called the *force of friction* or simply *friction*. This is a force which always acts opposite to the motion, and depends on the nature and condition of the surfaces in contact. A distinction is usually made between static friction and kinetic friction. *Static* friction is that which opposes a tendency to move due to the forces acting on a body at rest, and its limiting value occurs when the body is at the point of transition from rest to motion. *Kinetic* friction is that which continually opposes the motion. The amount of friction in both cases depends on the roughness and the hardness of the surfaces in contact, the kind and the amount of lubrication, and other conditions.

It should be remembered that by the term motion is meant the relative motion of the surfaces of contact. Thus a force of friction may act as a *resistance* to the motion of a weight sliding on a plane and, on the other hand, a force of friction exerted between a belt and a pulley may act to *accelerate* the motion of the pulley. In this case both the belt and pulley are in motion; and the friction may be either static or kinetic, depending on the relative motion of the surfaces in contact.

A distinction is sometimes made between the adhesion of two surfaces and the friction between them, the adhesion being said to depend on the nature and extent of the surfaces, but to be independent of the pressure. For very small pressures the adhesion would form a considerable portion of the resistance to motion, but with large pressures, the adhesion is relatively so small that it may be neglected. As it is impracticable to distinguish between adhesion and friction, the determination of all such resistances being entirely a matter of experiment, we include them all under the head of friction

As allowances for friction enter into nearly every engineering problem dealing with motion we will proceed with the consideration of certain fundamental relations between the friction and the other forces acting upon a body when at rest and when in motion.

180. Coefficient of Sliding Friction. — If a weight W is moved along a horizontal plane, by a horizontal force P (Fig. 187),

a force of friction will be developed at the surface between the weight and the plane. If we let F = the friction and N = the normal pressure between the plane and the weight, the ratio of F to N is called the *coefficient of friction.*

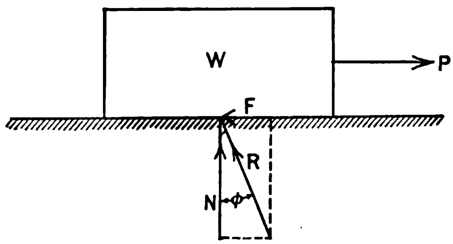

FIG. 187.

Denoting this coefficient by f, we have

$$f = \frac{F}{N}.$$

If we let R equal the resultant of F and N, and ϕ the angle which R makes with N, we shall have

$$R = \sqrt{F^2 + N^2},$$

and

$$f = \frac{F}{N} = \tan \phi;$$

that is, *the coefficient of sliding friction* is equal to the tangent of the angle ϕ between the resultant force exerted by the plane on the weight and its normal component.

The angle ϕ is called the *angle of friction* and the cone, generated by revolving R about the normal N, is called the *cone of friction.*

In the case shown in Fig. 187 it is evident that N is equal to W and, if the weight moves with a uniform velocity, P is equal to F and $f = \frac{F}{N} = \frac{P}{W}$ is the coefficient of kinetic friction. If the weight is at rest and the force P is the force required to start it in motion, $f = \frac{P}{W}$ is the coefficient of static friction. Generally the latter coefficient is larger than the former. In both cases the resultant of P and W, that is, the resultant reaction of the weight against the plane, will be equal and opposite to R.

If the magnitude of P is such that the line of action of the resultant of P and W falls within the cone of friction, no motion will result and, on the other hand, if the line of action of the resultant falls outside of the cone of friction, the body will move with an accelerated motion.

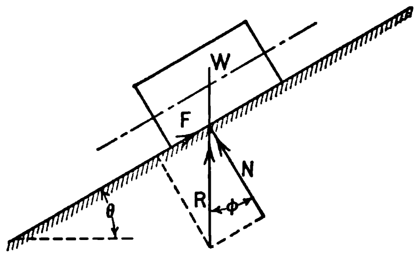

FIG. 188.

A special case is that of the weight W resting on an inclined plane (Fig. 188) under the action of gravity and the reaction R of the plane on the weight. It is evident that if θ, the angle which the plane makes with the horizontal, is less than ϕ, the angle of friction, no motion will result and, if θ is greater than ϕ, the weight will slide down the plane. When $\theta = \phi$ the weight will be at the point of sliding down the plane. For this reason ϕ is sometimes called the *angle of repose*. The coefficient $f = \tan \phi$ may be constant, or variable, for surfaces of the same material, depending on conditions.

181. Laws of Sliding Friction. — There are certain laws, based upon the results of experiments, which are known as the laws of friction, which may be stated as follows:

(1) Friction is proportional to the normal pressure between the rubbing surfaces.

(2) Friction is independent of the extent of the surfaces in contact.

(3) Friction is independent of the relative velocities of the sliding surfaces.

(4) Static friction is generally greater than kinetic friction.

These laws were originally based upon experiments made by Coulomb and Morin and within certain limits may be said to hold for the friction of non-lubricated surfaces. To them may be

added the following, based upon experiments on lubricated surfaces.

(5) The friction of lubricated surfaces is generally less than that of dry, or non-lubricated, surfaces and depends less upon the nature of the surfaces than upon the nature of the lubricant and the method of applying it.

The results of experiments made since those of Morin seem to show that the above laws of kinetic friction are true only for moderate speeds and moderate pressures. Experiments at slow speeds have shown that, in certain cases, the coefficient of friction at low speeds has been higher than the coefficient at moderate speeds. Also at very high speeds, under certain conditions, the coefficient of friction has been found to diminish as the speed increased and also appears to be affected by an increase in temperature of the sliding surfaces. Provided the pressure does not become great enough to crush the material, the coefficient of friction tends as a rule to diminish as the pressure increases.

In the case of lubricated surfaces, the coefficient will depend on the rate at which the lubricant is applied. When a bearing is flooded in oil, the friction apparently follows the laws of fluid friction more closely than those of solid friction.

According to the laws of fluid friction, the friction is independent of the intensity of pressure, is proportional to the surface of contact, and to the square of the speed, approximately.

It will be apparent, therefore, that in estimating the coefficient of friction in any case it is necessary to know the results of experiments made under similar conditions; and that it is impossible to formulate exact laws to cover a variety of conditions.

In tables based on the experiments on friction made by Morin and others will be found values for the coefficient of sliding friction between smooth dry metal surfaces ranging from 0.15 to 0.35, depending on the nature of the metal; while for the coefficients between smooth surfaces of wood and between wood and metal the values range from 0.25 to 0.60. If smooth metal surfaces are lubricated with grease or oil, the coefficients at moderate pressures range from values in the neighborhood of 0.15 to 0.05. In the case of well lubricated journals much lower values are obtained.

Certain cases dealing with static and kinetic friction will now

be discussed. In each case we will denote the coefficient of friction by $f = \tan \phi$.

182. The Weight on the Inclined Plane.— Let P represent the resultant of all the forces, except gravity and the reaction of the plane, acting on a weight W which rests on an inclined plane (Fig. 189), and let θ = the angle which the plane makes with the horizontal and α = the angle which P makes with the plane. Let N = the normal pressure of the plane on the weight and assume that P, N and W are in the same plane. Let F = the force of friction and R = the resultant of F and N.

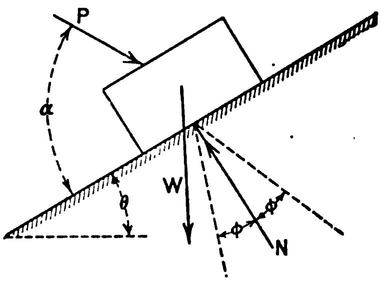

FIG. 189.

We will consider two cases: (a) when there is no motion; (b) when the weight slides up, or down, the plane.

(a) In this case the angle between R and N will be equal to or less than ϕ, and it is evident that R, P and W will pass through the same point. By resolving the forces into components perpendicular and parallel to the plane and applying the conditions of equilibrium (Art. 35), we obtain, when the weight is at the point of moving up the plane,

$$P \cos \alpha - W \sin \theta - F = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$-P \sin \alpha - W \cos \theta + N = 0; \quad . \quad . \quad . \quad . \quad . \quad (2)$$

also,

$$F = N \tan \phi \text{ (Art. 180)}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The solution of these equations will give P, F and N in terms of W.

If the weight were at the point of sliding down the plane, the force of friction would act in the opposite direction and the sign of F in equation (1) would be plus instead of minus.

(b) In this case, if the motion is uniform, the forces acting on the weight will be in equilibrium and equations (1), (2) and (3) will hold true as before. If the motion is accelerated, the vector sum of the components perpendicular to the plane will be equal to zero, and the resultant P_r, of all the forces acting on the weight will be equal to the vector sum of the components parallel to the plane.

Hence, when the motion is up the plane,

$$P \cos \alpha - W \sin \theta - F = P_r, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$-P \sin \alpha - W \cos \theta + N = 0 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and
$$F = N \tan \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The solution of these equations will give P_r, F and N in terms of P and W. If the motion were down the plane the sign of F in equation (4) would be plus instead of minus.

It is evident from equation (1) that the work done in moving the weight up the plane, with a uniform velocity through the distance s, will be equal to

$$P \cos \alpha \times s = Ws \sin \theta + Fs = Wh + Fs, \quad . \quad . \quad . \quad (7)$$

where $h = s \sin \theta$, the vertical projection of the distance s.

183. The Wedge. — Let the forces acting on the wedge ACB (Fig. 190) be parallel to a plane which is perpendicular to the edge

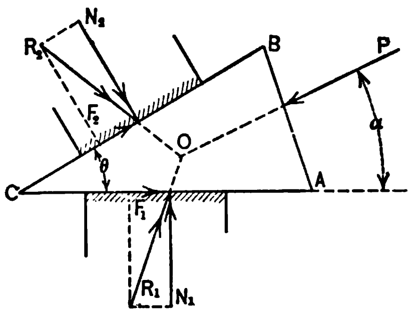

Fig. 190.

of the wedge at C. Let the force P, making an angle α with the side CA, be the force required to keep the wedge in uniform motion toward the left; and let f_1 = the coefficient of friction at the surface CA and f_2 = the coefficient of friction at the surface CB. Let F_1 = the friction and N_1 = the normal pressure on the surface CA; F_2 = the friction and N_2 = the normal pressure on the surface CB. These forces must be distributed in such a manner that their resultants, R_1 and R_2, and the force P will intersect at some point O. Let the angle $ACB = \theta$.

Resolving all the forces into components parallel and per-

pendicular to AC and writing the equations for equilibrium, we have

$$-P\cos\alpha + F_1 + F_2\cos\theta + N_2\sin\theta = 0, \quad . \quad . \quad (1)$$

and $$-P\sin\alpha + N_1 + F_2\sin\theta - N_2\cos\theta = 0; \quad . \quad (2)$$

also, $$F_1 = f_1N_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and $$F_2 = f_2N_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The solution of these equations will give the values of F_1, F_2, N_1 and N_2 when P, f_1 and f_2 are known; or the value of P, when N_2, or N_1, and f_1 and f_2 are known.

If the pressure P is applied along the center line of the wedge, bisecting the angle θ, and $f_1 = f_2$, it is evident that we shall have

$$N_1 = N_2 \text{ and } F_1 = F_2.$$

Hence, by resolving the forces into components parallel and perpendicular to the line of action of P and noting that $\alpha = \dfrac{\theta}{2}$, we obtain

$$P = 2\,F_1\cos\alpha + 2\,N_1\sin\alpha = 2\,N_1\,(f_1\cos\alpha + \sin\alpha). \quad . \quad (5)$$

184. Axle Friction. — If an axle fits loosely in a cylindrical bearing and both the axle and bearing are perfect cylinders, the bearing surface will be a narrow strip which may be regarded as a single element. Theoretically, if the cylinders were incompressible, they would be in contact along a single line. Let r = the radius of the shaft and CB represent the direction of the resultant pressure on the bearing (Fig. 191), due to the weight of the shaft, pulleys, etc., and pulls or thrusts, due to belting, gearing, or other causes.

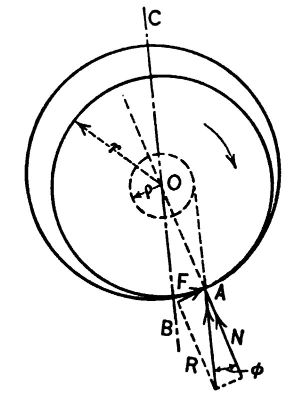

Fig. 191.

If the shaft turns in the direction of the arrow, it will tend to rise in the bearing until it rotates in contact with some element A. If N = the normal pressure and F = the friction exerted by the bearing on this element, their resultant

$$R = \sqrt{F^2 + N^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

must be equal and opposite to the resultant pressure on the bearing, and the moment of R about O, the center of the shaft, will

be equal to Fr, the moment of the friction. If ϕ = the angle of friction we shall have

$$F = N \tan\phi = R \sin\phi, \quad . \quad . \quad . \quad . \quad (2)$$

that is, the friction will equal the product of the resultant pressure on the bearing and the sine of the angle of friction.

If, with O as a center, a circle is drawn tangent to the line of action of R, its radius will be equal to

$$\rho = r \sin\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the moment of the friction will be equal to

$$M = Fr = R\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This circle is called the *friction circle*.

For a lubricated bearing $f = \tan\phi$ is very small, and $\tan\phi$ may be substituted for $\sin\phi$ in equations (2) and (3) giving

$$F = R \tan\phi = fR, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$\rho = r \tan\phi = fr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If the direction of rotation is changed, the resultant pressure R will evidently act in the direction of the tangent to the friction circle passing through the new point of contact.

It is evident from equation (4) that the work done in overcoming friction during each revolution of the shaft will be equal to

$$2\pi M = 2\pi Fr = 2\pi R\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

185. Friction of an Axle, Bearing at Two Elements. — Let r = the radius of the axle which rotates in contact with the bear-

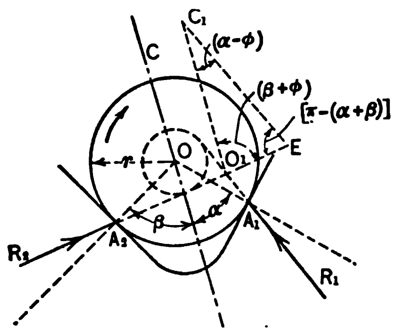

FIG. 192.

ing at the elements A_1 and A_2 (Fig. 192), and assume that the coefficient of friction at both A_1 and A_2 is equal to f. Let CO repre-

sent the direction of R, the resultant pressure on the bearing, and α = the angle between CO and the normal to the surface of contact at A_1, and β = the angle between CO and the normal at A_2. Assuming that the shaft turns in the direction of the arrow, let F_1 and F_2 equal the forces of friction, N_1 and N_2 equal the normal pressures, and $R_1 = \sqrt{F_1^2 + N_1^2}$ and $R_2 = \sqrt{F_2^2 + N_2^2}$ equal the resultant reactions of the bearing on the shaft at the points A_1 and A_2, respectively. Then R_1 and R_2 will be tangent to the friction circle and will make an angle ϕ with the radii OA_1 and OA_2, respectively. Their resultant will act through O_1, their point of intersection, and will be equal and parallel to R, and the moment of this resultant about O will be equal to the sum of the moments of F_1 and F_2.

From the triangle of forces whose sides C_1O_1, EC_1 and O_1E are equal to R, R_1 and R_2, respectively, we have

$$R_1 = R\frac{\sin(\beta + \phi)}{\sin(\alpha + \beta)}, \qquad R_2 = R\frac{\sin(\alpha - \phi)}{\sin(\alpha + \beta)}.$$

Hence

$$F_1 = R\frac{\sin(\beta + \phi)}{\sin(\alpha + \beta)}\sin\phi, \qquad F_2 = R\frac{\sin(\alpha - \phi)}{\sin(\alpha + \beta)}\sin\phi.$$

The moment of friction will then be equal to

$$M = F_1r + F_2r = \frac{Rr\sin\phi}{\sin(\alpha + \beta)}[\sin(\beta + \phi) + \sin(\alpha - \phi)]. \quad (1)$$

When $\alpha = \beta$ we have a special case and equation (1) reduces to

$$M = (F_1 + F_2)r = \frac{Rr\sin\phi}{\sin 2\alpha}(2\sin\alpha\cos\phi)$$

$$= \frac{Rr\sin\phi\cos\phi}{\cos\alpha} = R\rho\frac{\cos\phi}{\cos\alpha}. \quad (2)$$

Equation (2) shows that the moment of the friction increases as the angle α increases, being directly proportional to $\frac{1}{\cos\alpha}$.

When $\alpha = \beta = \phi$, the moment (equation 2) is the same as in the case where the shaft bears along a single element (Art. 184).

186. Friction of an Axle, Bearing Over a Surface. — Let r = the radius of the axle and let 2θ = the angle AOB, subtended by the arc of contact AB (Fig. 193) and assume the length of the bearing equal to unity.

Let p = the intensity of the normal pressure at any point in the bearing surface, and assume that p = a constant.

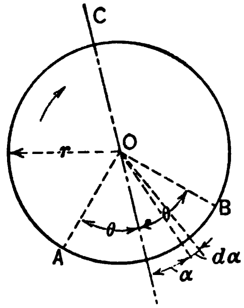

FIG. 193.

The normal pressure on an element of the surface, subtending an angle $d\alpha$, will be equal to $prd\alpha$ and its component parallel to OC will equal $pr \cos\alpha \, d\alpha$. Since the components, perpendicular to OC, of the pressure on any pair of elements on opposite sides of and equidistant from OC will balance, the resultant of the normal pressure on the bearing will be equal to

$$R = pr\int_{-\theta}^{\theta} \cos\alpha \, d\alpha = 2\, pr \sin\theta,$$

and hence

$$p = \frac{R}{2\, r \sin\theta}. \qquad (1)$$

The friction on the elementary arc $rd\alpha$ will be equal to

$$fpr\, d\alpha,$$

and its moment about O will be equal to

$$fpr^2\, d\alpha.$$

Hence the resultant moment of the friction will be equal to

$$M = fpr^2\int_{-\theta}^{\theta} d\alpha = 2\, fpr^2\, \theta. \qquad (2)$$

Substituting the value of p from equation (1) we have

$$M = \frac{fRr\theta}{\sin\theta} = \frac{R\rho\theta}{\sin\theta}, \qquad (3)$$

where R, the resultant of the normal pressure, may be assumed to be equal to the resultant pressure on the bearing surface, when f is a small quantity.

For small areas of contact, $\theta = \sin\theta$, nearly, and we may write

$$M = R\rho; \qquad (4)$$

that is, we may assume that the pressure acts at a single element as in Art. 184.

When the entire arc of contact is 60 degrees, θ = 30 degrees and the error in assuming $\theta = \sin\theta$ is less than 5 per cent.

If the arc of contact is 180 degrees,

$$M = \frac{fRr\pi}{2} = \frac{R\rho\pi}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In making the assumption in the above discussion that p = a constant, no account has been taken of the fact that, because of the friction, the shaft tends to bear more heavily on one side of the arc of contact than on the other, as illustrated in the case in Art. 185; and also that, as the shaft wears the bearing, the intensity of the normal pressure is not likely to remain uniform over the entire surface of contact.

Hence it is impracticable to go to the refinement of equation (3) in calculating the moment of the friction, the formula

$$M = fRr \text{ (Art. 184)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

being the one ordinarily used for all cases of shaft friction.

It should be noted that the intensity of the normal pressure on the bearing as given by equation (1), is equal to the resultant pressure divided by the projection of the bearing surface on a plane perpendicular to the direction of that pressure.

In most engineering computations the *pressure per unit of bearing area* is calculated by dividing the resultant pressure by the product of the length and diameter of the bearing. Such computations are made to determine the size of a bearing required to run smoothly, without heating, under given conditions, the maximum allowable intensity of pressure in any given case being determined from experiments and the results of practice.

187. Pin Friction. — Pin friction, as it occurs in linkwork and jointed frames, is similar to axle friction. Let A and B be two pins, joined together by a link or other rigid body, and let the pins A and B move relatively to the link in the directions indicated by the arrows (Fig. 194).

If the coefficients of friction are known, the friction circles at A and B can be drawn.

If the link is in tension we find, on comparing with the shaft (Art. 184) that the normal pressure and the friction on the pins will act in the directions indicated and the line of action of the resultant pull between the pins will be the line CD, tangent to the friction circles.

If the link were in compression, or the direction of the motion changed, the line of action of the resultant force acting between the pins could be determined in a similar manner. In any case the re-

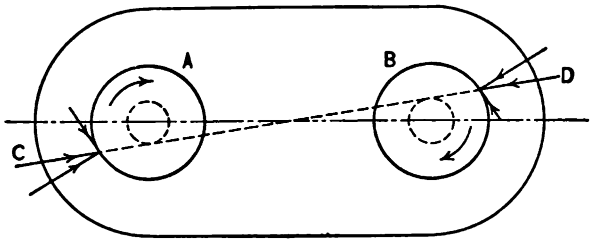

FIG. 194.

sultant pull or thrust would not act along a line connecting the centers of the pins, but at a small angle with it which would be equal to

$$\sin^{-1}\frac{\rho \pm \rho_1}{l},$$

where l = the length of the link between centers and ρ and ρ_1 equal the radii of the friction circles, ρ_1 being positive when the pins turn in the same direction relatively to the link and negative when they turn in opposite directions.

188. Pivot Friction. — When an axle, or shaft, is vertical or inclined, and sometimes in the case of a horizontal shaft, the end thrust is taken up by a support against which the axle, or shaft, bears as a pivot. The bearing may cover a part, or the entire end, of the shaft, or, it may support a collar on the shaft.

As the distribution of the friction is assumed to be the same whether the shaft is vertical or horizontal, we shall confine the following discussion to the vertical pivot.

Let Fig. 195 represent the cross section through the axis OY and the projection of the bearing surface of any pivot and let r_1 and r_2 be the radii of the circles at the inside and outside limits

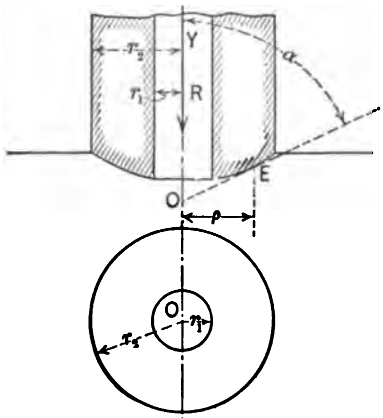

FIG. 195.

of the bearing surface. Let R = the resultant bearing pressure on the pivot and let p = the intensity of the bearing pressure at any point E, whose perpendicular distance from OY is equal to ρ, and let α = the angle between OY and the tangent to the bearing surface at E.

The normal pressure on an element dA of the bearing surface at the point E will be equal to

$$p\,dA.$$

The friction on this element will be equal to

$$fp\,dA,$$

and its moment about OY will be equal to

$$fp\rho\,dA.$$

Hence the resultant moment of the friction on the entire surface will be equal to

$$M = f\int p\rho\,dA. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

To integrate this expression we must know the form of the pivot and the expression for the value of p at any point.

If we resolve the elementary pressure $p\,dA$ into horizontal and vertical components, it is evident that the horizontal component will be balanced by that on the elementary area symmetrically situated on the other side of the axis and, since this will be true for every elementary area, the resultant pressure R will be equal to the sum of the vertical components of the elementary pressures. The vertical component of $p\,dA$ will be equal to

$$p \sin \alpha\,dA = pdA_1,$$

where $dA_1 = \sin \alpha\,dA$ is the projection of dA on the horizontal plane.

Therefore $$R = \int p \sin \alpha\,dA = \int p\,dA_1.$$

Expressing dA_1 in polar coördinates, we have

$$p\,dA_1 = p\rho\,d\rho\,d\theta,$$

and hence

$$R = \int p\,dA_1 = \int_{r_1}^{r_2} \int_0^{2\pi} p\rho\,d\rho\,d\theta. \quad . \quad . \quad . \quad (2)$$

Substituting in equation (1)

$$dA = \frac{dA_1}{\sin \alpha} = \frac{\rho\,d\rho\,d\theta}{\sin \alpha},$$

we have

$$M = f\int_{r_1}^{r_2}\int_0^{2\pi}\frac{p\rho^2\, d\rho\, d\theta}{\sin\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

To determine the value of p, one of two assumptions may be made; (a) that p = a constant; (b) that $\frac{p\rho}{\sin\alpha}$ = a constant. The latter is based on the assumptions that the amount of vertical wear at every point in the bearing surface is the same and that the wear at any point in the direction normal to the surface is proportional to the product of the intensity of pressure and the velocity of sliding at that point. Since the velocity at any point in the wearing surface is proportional to ρ, the wear at that point in the normal direction will be proportional to $p\rho$ and hence the wear in the vertical direction will be proportional to $\frac{p\rho}{\sin\alpha}$.

(a) If we assume p = a constant and let A = the horizontal projection of the total bearing area, we shall obtain from equation (2),

$$p = \frac{R}{A}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and from equation (3),

$$M = f\frac{R}{A}\int_{r_1}^{r_2}\int_0^{2\pi}\frac{\rho^2\, d\rho\, d\theta}{\sin\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

(b) If we assume $\frac{p\rho}{\sin\alpha} = C$ = a constant, and let A = the horizontal projection of the bearing area, as before, we shall obtain from equation (2),

$$C = \frac{R}{\int_{r_1}^{r_2}\int_0^{2\pi}\sin\alpha\, d\rho\, d\theta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and from equation (3),

$$M = fC\int_{r_1}^{r_2}\int_0^{2\pi}\rho\, d\rho\, d\theta = fCA. \quad . \quad . \quad . \quad (7)$$

From the last four equations the moment of the friction on any pivot may be determined on the basis of either of the above assumptions.

Expressions for the work done per revolution in overcoming friction may evidently be obtained by substituting the above values of M, in the formula

$$W = 2\pi M. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

189. Friction of Screw Threads. — Let W = the resultant pressure on the threads, acting parallel to the axis of the screw, h = the pitch of the screw and r = the mean of the inside and outside radii of the bearing surface of the thread.

Square Threaded Screw. — In this case the average angle of slope between the bearing surface of the thread and a plane perpendicular to the axis of the screw will be equal to

$$\theta = \tan^{-1}\frac{h}{2\pi r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If we let p_n = the average intensity of the normal pressure on the bearing surface, the friction per unit of area will be equal to

$$fp_n,$$

and the average moment of the friction per unit of area, about the axis of the screw, will be equal to

$$fp_n r \cos\theta \text{ (Art. 61).}$$

Hence, if we let A = the total area of the bearing surface of the thread, the total moment of the friction will be equal to

$$fp_n Ar \cos\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the work done per revolution in overcoming the friction on the thread will be equal to

$$2\pi fp_n Ar \cos\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The work done per revolution in raising the weight will be equal to

$$Wh,$$

and hence if we let M = the moment of the resultant couple applied to turn the screw, the work done per revolution in lifting the weight W and overcoming the friction will be equal to

$$2\pi M = 2\pi fp_n Ar \cos\theta + Wh. \quad . \quad . \quad . \quad . \quad (4)$$

Substituting the value of h (equation 1) and dividing by 2π we obtain

$$M = fp_n Ar \cos\theta + Wr \tan\theta. \quad . \quad . \quad . \quad . \quad (5)$$

In the ordinary single threaded screw the pitch is so small that we may assume, with a very small error, that

$$\cos\theta = 1 \quad \text{and} \quad p_n = \frac{W}{A},$$

in which case equation (4) reduces to

$$2\pi M = 2\pi Wfr + Wh, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and equation (5) to

$$M = Wfr + Wr\tan\theta = Wr(\tan\phi + \tan\theta). \quad . \quad . \quad (7)$$

If the weight W is being lowered by turning the screw the last term in each of the equations (4), (5), (6) and (7) will evidently be negative.

V *Threaded Screw.* — Let α = the angle between the surfaces of the thread, A = the total area of the bearing surface of the thread, and p_n = the average intensity of the normal pressure. Then $p_n = \dfrac{W}{A\cos\frac{\alpha}{2}}$, approximately; and by substituting this value in equations (4) and (5) we obtain for the work done per revolution in lifting the weight,

$$2\pi M = 2\pi f\frac{W}{\cos\frac{\alpha}{2}}r\cos\theta + Wh, \quad . \quad . \quad . \quad . \quad (8)$$

and, for the magnitude of the turning couple,

$$M = f\frac{Wr\cos\theta}{\cos\frac{\alpha}{2}} + Wr\tan\theta = Wr\left(\frac{\tan\phi\cos\theta}{\cos\frac{\alpha}{2}} + \tan\theta\right). \quad (9)$$

If $\alpha = 60°$ and we assume $\cos\theta = 1$, equation (9) becomes

$$M = Wr\,(1.15\tan\phi + \tan\theta). \quad . \quad . \quad . \quad . \quad (10)$$

If the weight is lowered by turning the screw, the last term in equations (8), (9) and (10) will be negative.

190. Friction Cones and Friction Discs. — For the purpose of transmitting power to a machine, where it is necessary to start and stop the machine independently of the driving mechanism, a friction clutch is frequently used. As the name indicates, the power is transmitted through the clutch by the force of friction exerted between surfaces which are brought into contact and subjected to pressure when the clutch is "closed."

In certain types of clutches the surfaces in contact are conical in form while in others the surfaces are simply flat discs. In either case the surfaces can be brought into contact and subjected to pressure by some form of mechanism which can be operated while the clutch is turning. In operating the clutch to start a machine the friction between the surfaces of contact is at first kinetic, as the surfaces slip by each other until the machine is brought up to speed and the clutch is closed. The friction then becomes static, unless the clutch is overloaded to the point where continual slipping occurs.

If the pressure on the surfaces of contact is assumed to be uniform in intensity, the formulas for the moment of friction in terms of the coefficient of friction and the resultant pressure will be identical in form with the formulas for pivot friction (Art. 188), the formula for the conical clutch being the one deduced in Problem 11, Art 198, and that for the flat disc in Problem 10, Art. 198.

If in the latter case, as is frequently the custom, more than one disc is used the total moment of the friction may be found by multiplying the formula for the moment of friction in Problem 10, Art.198 by n, the number of surfaces of contact.

191. Belt Friction. — In the transmission of power by belting from one pulley to another, the force acting between the belt and either pulley is the friction at the surface of contact.

In order to produce the normal pressure necessary to develop a sufficient amount of friction, it is necessary to tighten the belt to an initial tension, which will be practically equal on the two sides of the pulley when the pulley is at rest.

When power is being transmitted, the tension will be greater on one side of the pulley than on the other, and the difference of the tensions will be equal to the total force of friction at the surface of contact of the belt and the pulley. The ordinary method of deducing the relation between the two tensions and the coefficient of friction between the belt and the pulley is the following:

Let T_1 and T_2 be the tensions in a belt, transmitting power to a pulley of radius r (Fig. 196), and let $T_1 > T_2$. T_1 is called the tension in the *tight* side of the belt and T_2 the tension in the *loose* side. Let α = the angle subtended by the arc of contact AB, and p = the normal pressure per unit length of arc at any point C on the surface of the pulley.

The normal pressure on the elementary arc CC_1, subtending the angle $d\theta$, will be equal to $pr\,d\theta$.

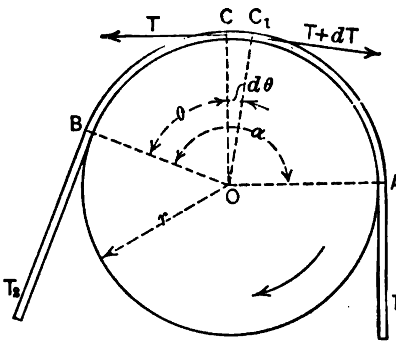

FIG. 196.

If the belt is at the point of slipping, and f = the coefficient of friction between the belt and pulley, the friction on the arc CC_1 will be equal to

$$fpr\,d\theta.$$

If we let T = the tension at C under these conditions, the tension at C_1 will equal $T + dT$, where

$$dT = fpr\,d\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Before integrating this expression we must determine the value of p in terms of T. Let CC_1 be a very small length of a flexible band which is wrapped around a cylinder, of radius r, and subjected to a uniform tension T (Fig. 197).

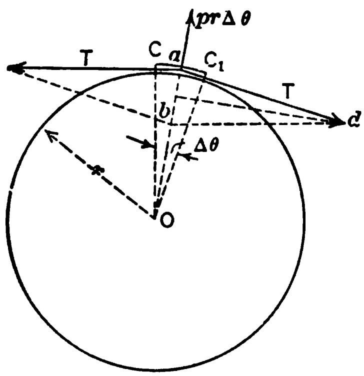

FIG. 197.

Let p = the pressure per unit of length between the band and the cylinder and let $\Delta\theta$ = the angle subtended at the center by the arc CC_1. The forces acting on CC_1 will be the two forces T, parallel to the tangents to the cylinder at C and C_1, and the resultant of the normal pressure between the band and the cylinder which will equal $pr\Delta\theta$, very nearly.

Since these forces are in equilibrium when the pulley is at rest, or rotating with a uniform speed, we have from the force triangle *abd*

$$\frac{pr\Delta\theta}{2} = T \sin\frac{\Delta\theta}{2},$$

and hence,

$$p = \frac{T\,2\sin\frac{\Delta\theta}{2}}{r\Delta\theta}, \text{ very nearly.}$$

At the limit, as $\Delta\theta$ diminishes, this expression becomes

$$p = \frac{T}{r}. \qquad \ldots\ldots \qquad (2)$$

If, for a small finite value of $\Delta\theta$, the tension at C_1 is equal to $T + \Delta T$, it is evident that as $\Delta\theta$ approaches zero ΔT also approaches zero and at the limit the value of p will be that given by equation (2) as before.

Substituting this value of p in equation (1) and transposing we have

$$\frac{dT}{T} = f\,d\theta.$$

Integrating between the limits of $T = T_1$ and $T = T_2$ and the corresponding limits of θ we have

$$\int_{T_2}^{T_1} \frac{dT}{T} = f\int_0^{\alpha} d\theta,$$

and

$$\log_e T_1 - \log_e T_2 = \log_e \frac{T_1}{T_2} = f\alpha,$$

or

$$\frac{T_1}{T_2} = e^{f\alpha}, \qquad \ldots\ldots \qquad (3)$$

where e = the Naperian base 2.718 +.

Equation (3) gives the theoretical relation between T_1 and T_2, when the belt is at the point of slipping on the pulley, in terms of the coefficient of friction between the belt and the pulley and the angle subtended by the arc of contact, when no allowance is made for the effect of centrifugal force.

If the belt is traveling at a high speed the centrifugal force will tend to diminish the normal pressure, and hence the force of friction, between the belt and the pulley. In other words, a certain

portion of the tension in the belt will be required to furnish the centripetal forces necessary to make the particles of the belt move in circular paths around the pulley, the remaining part producing the normal pressure between the belt and pulley, as shown by equation (2).

If we let v = the speed of the belt and w = the weight per unit of length, the centripetal force acting on the elementary length CC_1 (Fig. 197) will be equal to

$$\frac{mv^2}{r} = \frac{wr\Delta\theta}{gr}v^2 = \frac{w\Delta\theta}{g}v^2, \text{ very nearly (Art. 151).}$$

This force will act toward the center O and will be the resultant of two equal tensions at the sections C and C_1.

If we let t = the magnitude of these tensions we shall obtain

$$\frac{1}{2}\cdot\frac{w\Delta\theta v^2}{g} = t\sin\frac{\Delta\theta}{2}.$$

Hence,

$$t = \frac{wv^2}{g}\cdot\frac{\Delta\theta}{2\sin\frac{\Delta\theta}{2}}, \text{ very nearly,}$$

and at the limit, as $\Delta\theta$ diminishes,

$$t = \frac{wv^2}{g}. \qquad (4)$$

For a belt of uniform section and material, t will be the component of the tension at every section, required to produce the circular motion in the belt or, as it is commonly called, the tension due to centrifugal force. Hence if T_1 and T_2 are the total tensions in the tight and loose sides of the belt, respectively, equation (2), modified to include the effect of centrifugal force, should be written:

$$\frac{T_1 - t}{T_2 - t} = e^{f\alpha} \qquad (5)$$

If W = the work transmitted per second by the belt, we shall have

$$W = (T_1 - T_2)\,v, \qquad (6)$$

and hence the expression for the horse power transmitted will be

$$\text{h.p.} = \frac{(T_1 - T_2)\,v}{550}. \qquad (7)$$

The solution of equations (5) and (7) will give the power which a

belt will transmit at any given speed, for any assumed values of T_1 and f, when the weight of the belt and the arc of contact between the belt and the pulley are known. In determining the tension due to centrifugal force the weight of a section of double leather belt 1 in. wide and 12 ins. long may be taken as $w = 0.15$ lbs.

A conservative estimate of the power which a double leather belt will transmit may be made by assuming $f = 0.3$ and $T_1 = 150$ lbs. per inch of width.

For determining the sizes of belt required to transmit small amounts of power when the arc of contact is approximately 180°, equation (7) is frequently given in the form,

$$\text{h.p.} = \frac{bV}{k}, \quad \ldots \ldots \ldots \ldots (8)$$

where b = the width of the belt in inches, V = the speed in feet per minute and k = a constant. In the case of a single leather belt different rules give values of k varying from 600 to 1000; and for a double leather belt, values varying from 350 to 560. In formula (8) no account is taken of centrifugal force and hence for high speeds the more conservative rules call for the larger values of k.

192. Rope Drives. — Where hemp or cotton ropes are used for the transmission of power it is customary to run the ropes over grooved pulleys, the cross sections of the grooves being of the general form shown in Fig. 198. The size of the grooves is such that the rope touches the sides, as indicated in the figure, but not the bottom.

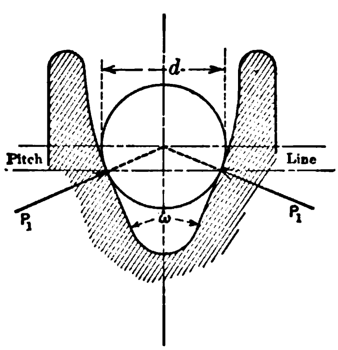

FIG. 198.

Let p_1 = the normal pressure per unit of length of rope, exerted on the side of the groove, f = the coefficient of friction between the rope and the pulley and r = the radius of the pitch line.

Then the resultant of the pressures p_1, on the two sides of the groove, will be a force

$$p = 2\,p_1 \sin\frac{\omega}{2},$$

acting in the radial direction.

Proceeding in the same manner as in Art. 191, the friction between the rope and the pulley on an arc of contact $rd\theta$ will be equal to

$$2\,fp_1r\,d\theta = \frac{fpr\,d\theta}{\sin\frac{\omega}{2}} = \frac{fT\,d\theta}{\sin\frac{\omega}{2}} = dT. \quad \ldots \quad (1)$$

Hence

$$\frac{dT}{T} = \frac{f}{\sin\frac{\omega}{2}}\,d\theta$$

and by integration we obtain

$$\frac{T_1}{T_2} = e^{\frac{f\alpha}{\sin\frac{\omega}{2}}}. \quad \ldots \quad (2)$$

In the form of groove most commonly used $\omega = 45°$ and equation (2) becomes

$$\frac{T_1}{T_2} = e^{2.61\,f\alpha}. \quad \ldots \quad (3)$$

To allow for the effect of centrifugal force, equation (3) should be written

$$\frac{T_1 - t}{T_2 - t} = e^{2.61\,f\alpha}, \quad \ldots \quad (4)$$

where $t = \frac{wv^2}{g}$, w = the weight per unit length and v = the speed of the rope.

The formula for the horse power transmitted will be

$$\text{h.p.} = \frac{(T_1 - T_2)\,v}{550} \text{ (Art. 191)}. \quad \ldots \quad (5)$$

In determining the tension in the rope due to centrifugal force the weight per ft. of Manila hemp rope may be taken as

$$w = 0.34\,d^2,$$

where d = the diameter of the rope in inches.

A conservative estimate of the power which a Manila rope will transmit may be made by assuming $T_1 = 200\,d^2$ and $f = 0.13$ in the above formulas.

When the arc of contact is 180°, by substituting this value of f in equation (4) and eliminating T_2 between equations (4) and

(5), a commonly accepted formula for the power which may be transmitted by a Manila rope is obtained, namely,

$$\text{h.p.} = \frac{\frac{2}{3}(T_1 - t)\,v}{550}, \quad \ldots \ldots \ldots \quad (6)$$

where $T_1 = 200\, d^2$ and $t = \dfrac{0.34\, d^2 v^2}{g}$, as indicated above.

193. Maximum Power which can be Transmitted by a Belt or Rope. — The formula for the ratio of the tensions (Arts. 191 and 192) in a belt, or rope, which is transmitting power, may be written

$$\frac{T_1 - t}{T_2 - t} = e^{c f \alpha}, \quad \ldots \ldots \ldots \quad (1)$$

where c = a constant, which is equal to unity for a flat belt and whose value depends on the slope of the sides of the groove in the case of a rope running on a grooved pulley.

The formula for the work transmitted per second by a belt or rope may be written

$$W = (T_1 - T_2)\, v. \quad \ldots \ldots \ldots \quad (2)$$

By transforming equation (1) we obtain

$$\frac{T_1 - T_2}{T_1 - t} = \frac{e^{c f \alpha} - 1}{e^{c f \alpha}} = 1 - e^{-c f \alpha}, \quad \ldots \ldots \quad (3)$$

and substituting in equation (2) the value of $(T_1 - T_2)$ from equation (3) we have

$$W = (T_1 - T_2)\, v = (1 - e^{-c f \alpha})\,(T_1 - t)\, v = (1 - e^{-c f \alpha})\left(T_1 - \frac{w v^2}{g}\right) v. \quad . \quad (4)$$

Differentiating equation (4) and placing the derivative equal to zero, we obtain

$$\frac{dW}{dv} = (1 - e^{-c f \alpha})\left(T_1 - \frac{3\, w v^2}{g}\right) = 0, \quad \ldots \ldots \quad (5)$$

and

$$v = \sqrt{\frac{g T_1}{3\, w}} = 3.27 \sqrt{\frac{T_1}{w}}. \quad \ldots \ldots \ldots \quad (6)$$

Hence the speed at which the maximum power will be transmitted by a belt, or rope, depends only on the maximum allowable tension and the weight per unit of length.

From equation (4) we may obtain the expression for the horse power transmitted

$$\text{h.p.} = \frac{1 - e^{-c/\alpha}}{550}\left(T_1 - \frac{wv^2}{g}\right)v, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and if a graph of this equation is made, with values of v as abscissæ and of the horse power as ordinates, it will be found that, beginning with low speeds, the power will increase as the speed increases up to a maximum value, beyond which a further increase in speed will result in a decrease in power until, if the speed is increased to the value at which the tension due to centrifugal force becomes equal to the allowable tension in the belt, the value of h. p. becomes again equal to zero.

Effect of Rigidity. — In the discussion in this, and the two preceding articles, no account has been taken of the friction losses due to the rigidity of the rope, or belt. No rope or belt is perfectly flexible, and in bending around a pulley a certain amount of work is done in overcoming the internal friction between the fibers. The amount of work lost in this manner will depend on the elasticity of the belt, or rope, as a whole, and the ratio of its thickness to the diameter of the pulley; the loss increasing as this ratio increases

The formula

$$\text{h.p.} = \frac{(T_1 - T_2)\,v}{550}$$

will not, therefore, give the exact amount of power which can be transmitted from one pulley to another; as a part of the difference in tensions will be required to perform the work necessary to overcome the rigidity.

The determination of this loss must be based entirely on experiment and no general rule can be given which will apply under all conditions.

194. Rolling Friction. — When one body rolls over the surface of another a resistance to the motion is developed which is called *rolling friction.* This in general is a small quantity, compared with sliding friction, and has been found to depend on the hardness of the surfaces in contact and the radius of the rolling surface.

The theory of rolling friction is based on the assumption that the bearing surfaces undergo a small amount of compression at the place of contact, and hence the effect is the same as if the roller

were continually being moved out of a hollow. Let P = the horizontal force which, when applied at the center O, is necessary to move the roller (Fig. 199) with a uniform speed along a horizontal

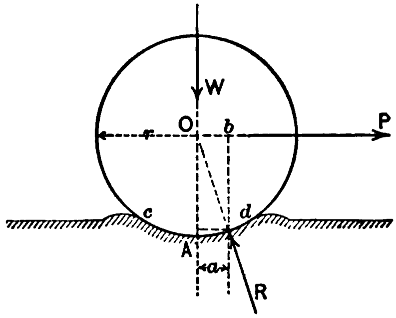

FIG. 199.

plane. Let W = the weight of the roller and R = the resultant pressure of the bearing surface on the roller. The force R will act at some point d in the bearing surface, whose perpendicular distance from the vertical OA we will call a, and its line of action will pass through O. Since the cross section of the bearing surface cd is distorted from the arc of a circle, R is not necessarily the resultant of the normal components of the prèssure on the different elements of the surface, but will be the resultant of the normal pressure and the friction, caused by the small amount of sliding which may take place during the distortion of the surfaces in contact. The amount of this sliding will depend on the nature of the surface, being considerable if either the roller or the bearing surface is very soft, and being imperceptible when both are hard. In either case, the resistance due to the sliding is included as a part of the rolling friction.

When the roller is moving at a uniform speed, the sum of the moments of the forces acting upon it, about any axis perpendicular to their plane, will be equal to zero and, if we take moments about an axis through the point d, we shall have

$$P\,(bd) = Wa.$$

But $$bd = r, \text{ very nearly.}$$

Therefore $$P = \frac{a}{r}W,$$

where a is the *coefficient of rolling friction.* Since a represents the length of a line, its value is expressed in units of length. For any given value of a, the rolling friction will therefore be directly proportional to the pressure on the bearing surface, and inversely to the radius of the roller.

The following values of a, determined by different experimenters, will give an idea of the magnitude of rolling friction for hard surfaces.

Lignum vitæ rollers on oak track (Coulomb) . . a = 0.0189 in.
Cast-iron wheels 20-inch diameter on cast-iron rails (Weisbach) . . . . a = 0.0183 in.
Cast-iron wheels 20-inch diameter on cast-iron rails (Rittinger) . . . . a = 0.0193 in.
Iron railroad wheels 30.4-inch diameter (Pambour) . . . . a = 0.0196 in. to 0.0216 in.

Some experimenters have found a to be a constant for rollers of different radii of the same material, and others have found it to vary.

If a is not constant it will be necessary to know its value for different sizes of rollers, as well as for different materials.

The quantity $\frac{a}{r}$ is evidently a ratio similar to the coefficient of sliding friction and, if we denote it as such, the formulas for rolling friction reduce to the same form as those for sliding friction.

195. Weight on Rollers. — If W is a weight, resting on rollers on a horizontal plane (Fig. 200), and r = the radius of the rollers,

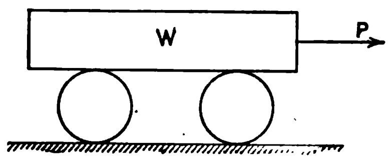

Fig. 200.

and a and a_1 are the coefficients of rolling friction at the two bearing surfaces, the force F necessary to move the weight with a uniform speed, if its line of action were along the line of centers

of the rollers, would be equal to the sum of the rolling frictions (Art. 194), namely,

$$F = \frac{a}{r} W + \frac{a_1}{r} W = \frac{W}{r} (a + a_1).$$

Since the speed of the weight is always twice that of the centers of the rollers the horizontal force, which must be applied to the weight to produce the motion, will be equal to

$$P = \frac{F}{2} = \frac{W}{2r} (a + a_1), \quad \ldots \ldots \quad (1)$$

and in the case where $a_1 = a$,

$$P = \frac{a}{r} W \quad \ldots \ldots \ldots \quad (2)$$

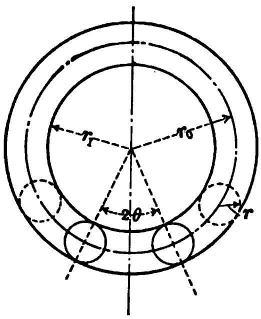

FIG. 201.

196. Roller Bearing. — Let r_1 = the radius of the shaft and r = the radius of the rollers (Fig. 201), and let W = the resultant pressure on the bearing. Let a and a_1 equal the coefficients of rolling friction on the inside and outside bearing surfaces, respectively.

(a) If we assume that the entire pressure W is carried on one roller, the force necessary to overcome the friction, if it were applied at the center of the roller, would be the same as in the case given in Art. 195, and hence would be equal to

$$P = \frac{W}{r} (a + a_1). \quad \ldots \ldots \ldots \quad (1)$$

Therefore, the moment of the friction about the center of the shaft will be equal to

$$M = P (r_1 + r) = \frac{W (a + a_1)}{r} r_0, \quad \ldots \ldots \quad (2)$$

where $r_0 = r_1 + r$, the distance from the center of the roller to the center of the shaft.

If $a_1 = a$, equation (2) becomes

$$M = 2 Wa \frac{r_0}{r}. \quad \ldots \ldots \ldots \quad (3)$$

(b) If we assume that the pressure is divided equally between two rollers, whose centers are on radii making the angle $2\,\theta$ with each other (Fig. 201), the pressure on each roller will be equal to

$$W_1 = \frac{W}{2 \cos \theta}$$

and the total moment of the friction of the two rollers will be equal to

$$M = \frac{2\,W_1\,(a + a_1)}{r}\,r_0 = \frac{W\,(a + a_1)}{r \cos \theta}\,r_0. \quad . \quad . \quad (4)$$

If $\qquad a_1 = a$, equation (4) becomes

$$M = 2\,Wa\,\frac{r_0}{r \cos \theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

(c) If the pressure is distributed over more than two rollers, it is necessary to make an assumption in regard to its distribution in order to determine the total moment of the friction. For example, if we assume that the pressure on each roller is the same, then, when the positions of the rollers are known, we can determine by the methods of statics the magnitude of the pressure on each one, in terms of the total pressure on the bearing. If the pressure is distributed over n rollers and we let w = the pressure on one roller, the total moment of the friction about the axis of the shaft will be equal to

$$M = nw\,\frac{(a + a_1)}{r}\,r_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If $a_1 = a$,

$$M = 2\,nwa\,\frac{r_0}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If the shaft bears on a few rollers which are near together,

$$nw = W, \text{ very nearly,}$$

and equations (6) and (7) become the same as equations (2) and (3).

If, on the other hand, the rollers are near together and bear equally over an arc of 180°, we shall have

$$nw = \frac{W}{2\,r}\,\pi r = \frac{W\pi}{2}, \text{ very nearly,}$$

where $\frac{W}{2\,r}$ = the intensity of pressure on the bearing surface, assuming it to be uniformly distributed (equation 1 Art. 186).

Substituting in equation (6) we have

$$M = \frac{W\pi\,(a + a_1)}{2\,r}\,r_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

If $a_1 = a$ equation (8) becomes

$$M = \pi W a \frac{r_0}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

197. Mechanical Efficiency. — In any mechanical device, or machine, for transmitting power a certain amount of work is done in overcoming friction in its various forms.

Therefore the *output* of work by the machine during any interval of time, at the beginning and end of which the energy of the moving parts of the machine is the same, is always less than the *input*, or the work required to run the machine during that time.

The ratio of the output to the input is the *mechanical efficiency*. This quantity is usually expressed in per cent and hence, if W_i = the input and W_o = the output of a machine during the interval of time indicated above, its efficiency will be equal to

$$e = \frac{W_o}{W_i} \times 100.$$

The quantity $W_i - W_o$ will represent the work done in overcoming friction, or, as it is frequently called the loss of energy due to friction, or, simply, the loss in the machine.

198. Problems. — Friction. — In the following problems the coefficient of friction will be denoted by the letter f in each case.

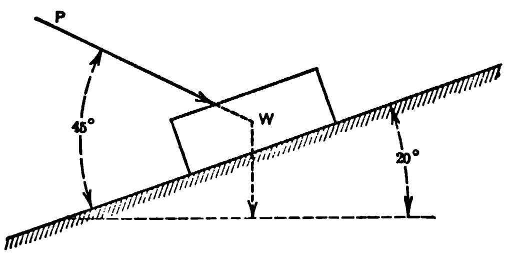

FIG. 202.

Problem 1.

Find the magnitude of the force P necessary to slide a weight W = 100 lbs. up the inclined plane (Fig. 202) at a uniform speed. Assume f = 0.2 and that the line of action P makes a constant angle of 45° with the plane. Find the work done in moving the weight through a distance of 10 ft.

Problem 2.

Find the magnitude of the force P necessary to prevent the weight in Problem 1 from sliding down the plane.

Problem 3.

If $f = 0.6$, find the magnitude and direction of the force P necessary to slide a weight of 100 lbs. down the plane at a uniform speed, assuming that the line of action of the force makes a constant angle of 45° with the plane as shown in Fig. 202.

Problem 4.

Solve Problem 3 (Art. 178), assuming that $f = 0.2$ and is a constant quantity.

Problem 5.

Solve Problem 5 (Art. 178), assuming that $f = 0.3$ and is a constant quantity.

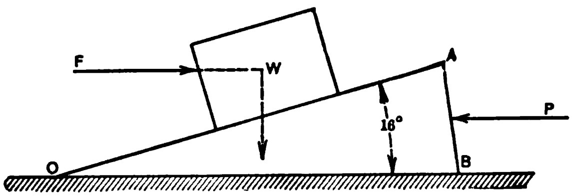

FIG. 203.

Problem 6.

The weight $W = 2000$ lbs. is raised by means of the wedge AOB which is moved by the horizontal force P (Fig. 203). The face OB of the wedge is horizontal and the weight is constrained to move in the vertical direction by a horizontal force F. If $f = 0.2$ for the face OB and $f = 0.3$ for the face OA find the magnitudes of P and F.

Problem 7.

Find the magnitudes of P and F in Problem 6, if $f = 0.25$ for both faces of the wedge and the force F is parallel to OA and the line of action of P bisects the angle AOB.

Problem 8.

A square threaded jackscrew of 3 in. outside diameter is turned by a bar 4 ft. long. If the pitch of the screw is 2 threads per inch and the bearing surface of the threads is $\frac{1}{4}$ in. wide, estimate the weight that can be lifted by a force of 50 lbs. applied at the end of the bar, neglecting the friction on the end of the screw. Assume $f = 0.10$.

Problem 9.

A journal is 4 ins. diameter and 6 ins. long. If the maximum allowable bearing pressure is 400 lbs. per sq. in. on the projection of the bearing area (Art. 186) and $f = 0.01$, find the moment of the friction on the bearing by the approximate formula. Find the work lost per revolution in overcoming the friction.

Problem 10. — Flat Circular Pivot.

Deduce the formulas for the moment of friction for the hollow flat pivot; for the solid flat pivot.

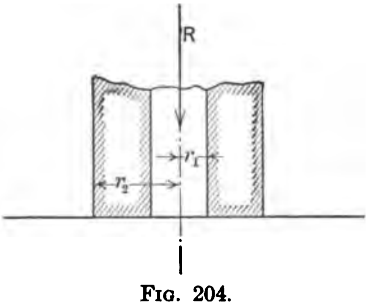

FIG. 204.

Solution. — Let r_2 = the outside radius and r_1 = the inside radius of the pivot (Fig. 204) and R = the total bearing pressure.

The formulas for the moment of friction, based on each of the two assumptions of the theory, may be deduced by substitution in the general equations (Art. 188) as follows:

Assumption (a): Substituting in equation (5) we obtain for the moment of the friction,

$$M = \frac{fR}{\pi (r_2^2 - r_1^2)} \times \frac{2\pi (r_2^3 - r_1^3)}{3} = \frac{2}{3} fR \frac{(r_2^3 - r_1^3)}{(r_2^2 - r_1^2)}. \quad . \quad . \quad . \quad (1)$$

Assumption (b): Substituting in equation (7) we obtain for the moment of friction,

$$M = \frac{fR\pi (r_2^2 - r_1^2)}{2\pi (r_2 - r_1)} = \frac{1}{2} fR (r_2 + r_1). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the pivot is solid, $r_1 = 0$ and, if we let $r_2 = r$, equations (1) and (2) reduce to

$$M = \frac{2}{3} f Rr, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$M = \frac{1}{2} f Rr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Problem 11. — Conical Pivot.

Deduce the formulas for the moment of friction for the conical pivot; for the same when the bearing extends to the apex of the cone.

Solution. — Let R = the total bearing pressure and let r_2 = the larger radius and r_1 = the smaller radius of the bearing (Fig. 205).

The formulas for the moment of friction, based on each of the two assumptions of the theory, may be deduced by substitution in the general equations (Art. 188) as follows:

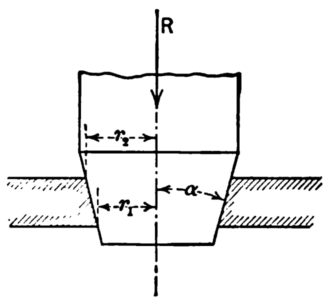

FIG. 205.

Assumption (a): Substituting in equation (5) the moment of the friction will be equal to

$$M = \frac{fR}{\pi (r_2^2 - r_1^2) \sin \alpha} \times \frac{2}{3} \pi (r_2^3 - r_1^3) = \frac{2}{3} \frac{fR}{\sin \alpha} \frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}. \quad . \quad (1)$$

Assumption (*b*): Substituting in equation (7) the moment of the friction will be equal to

$$M = \frac{fR\pi\,(r_2^2 - r_1^2)}{\sin\alpha\, 2\pi\,(r_2 - r_1)} = \frac{fR\,(r_2 + r_1)}{2\sin\alpha}. \quad \ldots \ldots \quad (2)$$

For a complete cone $r_1 = 0$ and, if we let $r_2 = r$, equations (1) and (2) will reduce to

$$M = \frac{2}{3}\cdot\frac{fRr}{\sin\alpha}, \quad \ldots \ldots \quad (3)$$

and

$$M = \frac{1}{2}\cdot\frac{fRr}{\sin\alpha}. \quad \ldots \ldots \quad (4)$$

Problem 12. — Spherical Pivot.

Deduce the formulas for the moment of friction for the spherical pivot, when the bearing surface is a segment less than the hemisphere; when the bearing surface is hemispherical.

Solution. — Let R = the total bearing pressure, r = the radius of the sphere and ϕ = the half angle subtended by the bearing surface (Fig. 206).

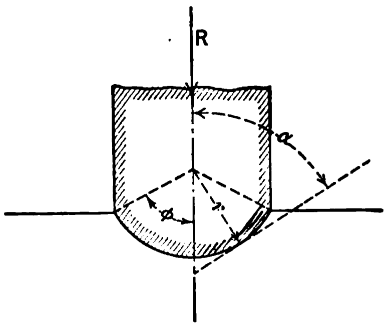

FIG. 206.

Let α = the angle between the axis of the pivot and the tangent to any point in the bearing surface at a distance ρ from the axis. Then

$$\rho = r\cos\alpha$$

$$\text{and } d\rho = -\,r\sin\alpha\, d\alpha,$$

and the formulas for the moment of friction, based on each of the two assumptions of the theory, may be deduced by substitution in the general equations (Art. 188) as follows:

Assumption (*a*): Substituting in equation (5) we obtain

$$M = -\frac{fRr^3}{A}\int_{\frac{\pi}{2}}^{\frac{\pi}{2}-\phi}\int_0^{2\pi}\cos^2\alpha\, d\alpha\, d\theta$$

$$= -\frac{fRr^3}{\pi r^2\sin^2\phi}\,2\pi\left[\frac{\alpha}{2} + \frac{\sin 2\alpha}{4}\right]_{\frac{\pi}{2}}^{\frac{\pi}{2}-\phi}$$

$$= fRr\frac{\phi - \sin\phi\cos\phi}{\sin^2\phi}. \quad \ldots \ldots \quad (1)$$

Assumption (*b*): Substituting in equation (7) we obtain

$$M = fCA = \frac{fRA}{-r\int_{\frac{\pi}{2}}^{\frac{\pi}{2}-\phi}\int_{0}^{2\pi}\sin^2\alpha\, d\alpha\, d\theta}$$

$$= -\frac{fR\, 2\pi r^2 \sin^2\phi}{2\pi r\left[\frac{\alpha}{2} - \frac{\sin 2\alpha}{4}\right]_{\frac{\pi}{2}}^{\frac{\pi}{2}-\phi}}$$

$$= fRr\frac{\sin^2\phi}{\phi + \sin\phi\cos\phi}. \qquad (2)$$

For the hemispherical pivot $\phi = 90°$, and equations (1) and (2) reduce to

$$M = \frac{\pi}{2} fRr, \qquad (3)$$

and

$$M = \frac{2}{\pi} fRr. \qquad (4)$$

Problem 13. — " Anti-Friction " Pivot.

In Schiele's so-called "anti-friction" pivot the bearing surface is so designed that both the first and second assumptions of the theory will hold. Such pivots do not eliminate friction but are supposed to wear uniformly and run smoothly at high speeds. The problem is to determine the form of a longitudinal section and the moment of the friction on the pivot.

Solution. — Since both the assumptions (*a*) and (*b*), namely $p =$ a constant and $\frac{p\rho}{\sin\alpha} =$ a constant, will hold in this case, it will follow that $\frac{\rho}{\sin\alpha} = b =$ a constant.

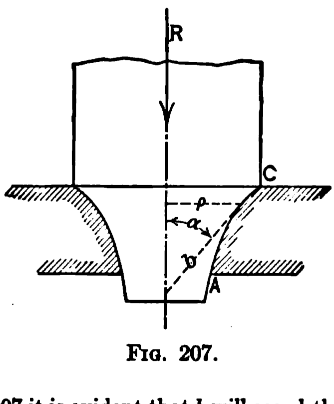

Fig. 207.

Referring to Fig. 207 it is evident that *b* will equal the length of the tangent at any point in the cross section of the bearing, from the point of tangency to the axis of the pivot. The curve described in this way is called the *tractrix*

and the surface of the pivot will be the surface of revolution described by revolving the tractrix around the axis.

If we let R = the total bearing pressure, we shall have, since p = a constant,

$$p = \frac{R}{A} \text{ (Art. 188)}, \quad \ldots \ldots \ldots \quad (1)$$

and since
$$\frac{\rho}{\sin \alpha} = b,$$

$$C = \frac{p\rho}{\sin \alpha} = pb = \frac{Rb}{A}. \quad \ldots \ldots \ldots \quad (2)$$

Substituting these values in equation (7) (Art. 188) we shall have

$$M = fCA = fRb. \quad \ldots \ldots \ldots \quad (3)$$

Therefore, the moment of the friction is proportional to the length of the tangent to the tractrix and is independent of the extent of the bearing surface.

Problem 14.

A step bearing, consisting of a flat circular pivot 4 ins. diameter, revolves at 600 revolutions per minute under a load of 2000 lbs. If f = 0.01 find the work lost per minute in overcoming friction by each assumption of the theory (Problem 10).

Problem 15.

Find the moment of the friction on a collar bearing, 4 ins. outside radius and 3 ins. inside radius, when subjected to a total pressure of 4000 lbs. Assume f = 0.03 and calculate the result by each of the assumptions of the theory (Problem 10). Find the work lost per minute in overcoming friction, if the speed is 300 revolutions per minute.

Problem 16.

A rule for determining the power which can be carried by a single leather belt is the following: A belt 1 in. wide, traveling at a speed of 1000 ft. per minute, will carry one horse-power (Art. 191). If α = 180° and f = 0.25, find T_1 and T_2 and the effective pull on the belt, neglecting the effect of centrifugal force. If the rule for a double leather belt 1 in. wide is "550 ft. per minute per horse-power" and α = 180° and f = 0.25, find T_1 and T_2.

Problem 17.

How many horse-power will a belt 6 in. wide carry when traveling at a speed of 4000 ft. per minute, allowing for the effect of centrifugal force. Assume that the weight of a strip of belt 1 in. wide and 12 ins. long is equal to 0.15 lb., and that T_1 = 150 lbs. per inch of width, α = 180°, f = 0.3.

Problem 18.

Assuming that the belt in Problem 17 runs on two pulleys of the same diameter, weighing 200 lbs. each, on two shafts in the same horizontal plane, find the resultant pressures on the bearings of the two shafts and estimate the total loss of power per minute between the two shafts due to friction at the bearings. Assume that the coefficient of shafting friction = 0.008. Diameter of each pulley = 4 ft. Diameter of each shaft = 4 ins.

Problem 19.

Find the width of belt necessary to carry 50 h.p. at a speed of 5000 ft. per minute, allowing for the effect of centrifugal force. Assume $T_1 = 150$ lbs. per inch of width, $\alpha = 160°$, $f = 0.25$, and that the weight of the belt is 0.15 lb. for a section 1 in. wide and 12 ins. long. If the driving shaft turns at 400 revolutions per minute find the diameter of the pulley required on that shaft.

Problem 20.

Find the maximum power which the 6 in. belt in problem 17 will transmit, assuming the values for α, f, w, and T_1 there given. (See Art. 193.)

Problem 21.

Find the maximum horse-power which a belt 8 ins. wide will transmit, if $\alpha = 180°$ and $w = 0.15$ lbs. for a section 1 in. wide and 12 ins. long; and the allowable values for T_1 and f are $T_1 = 150$ lbs. per inch of width, $f = 0.26$. Find the velocity of the belt and T_2.

Problem 22.

Find the power which a Manila hemp rope $1\frac{1}{4}$ ins. diameter will transmit when traveling at a speed of 4000 ft. per min., assuming $\omega = 45°$, $T_1 = 200\, d^2$, $f = 0.13$ and an arc of contact of 180° (Art. 192).

Problem 23.

Find the speed of the rope (problem 22) at which the maximum power will be transmitted, assuming the constants as there given; also the maximum power and the value of T_2. (See Art. 193).

Problem 24.

Solve Problem 22 by using formula (6) (Art. 192).

Problem 25.

With values of speed as abscissæ and horse-power as ordinates, plot the diagram for the power which can be transmitted at different speeds by a double leather belt 1 in. wide, assuming that the belt weighs 0.15 lb. per ft. of length, $\alpha = 180°$, $T_1 = 150$ lbs. and $f = 0.3$ (Art. 193).

Problem 26.

With values of speed as abscissæ and horse-power as ordinates, plot the diagram for the power which can be transmitted by a Manila hemp rope, 1 in. diameter, when running at different speeds on pulleys with 45° grooves, assuming $w = 0.34d^2$, $T_1 = 200d^2$ and $f = 0.13$ (Art. 193).

Problem 27.

A horizontal force of 100 lbs. is required to draw a weight of 4 tons, resting on rollers 6 ins. diameter, along a horizontal plane. Find the coefficient of rolling friction, assuming it to be the same at the under surface of the weight and at the surface of the plane.

Problem 28.

If the shaft in a roller bearing is 4 ins. diameter and the rollers $\frac{1}{2}$ in. diameter and the weight on the bearing is 5 tons, find the loss in work per revolution due to friction in the bearing; (*a*) assuming that the entire weight is carried on one roller; (*b*) assuming it to be uniformly distributed half way around the bearing. In each case assume the coefficient of rolling friction = 0.02 (Art. 196).

§ 4. Kinetics of Rigid Bodies Having Plane Motion Only.

199. Motion of a Rigid Body. — When a motion of any kind is imparted to a body, the resultant of the system of forces acting upon it may be determined by finding the resultant of the system of forces necessary to impart the required motion to each of the particles (Art. 141) of which the body is composed.

If the change in the relative positions of the particles of a body under the action of a system of external forces is so small as to be negligible, the body may be treated as if it were rigid (Art. 25). When the motion of such a body is known, the velocity and acceleration of each one of its particles can be determined, and the normal and tangential components of the resultant force acting upon each particle can be found (Art. 150). Then, as stated above, the resultant force, or system of forces, required to impart the motion to the entire body will be the resultant of the system comprising the forces acting on the individual particles into which the body is conceived to be divided.

200. Translation and Rotation. — When the linear velocities of all the particles of a rigid body are the same at each instant, the body is said to have a *motion of translation.* The motion of translation may be *rectilinear*, or *curvilinear*, and it may be *uniform* or *variable.* The paths of all the particles will be identical in form and a straight line joining any two particles will always have the same direction.

When, at each instant, the linear velocities of all the particles of a rigid body are proportional to their perpendicular distances from a given line, the body is said to have a *motion of rotation* and the line is called the *axis of rotation.* The motion of rotation may be *uniform*, or *variable*, and the position of the axis of rotation may be *fixed*, or *variable.* When the position of the axis is variable, the line about which the body rotates at any instant is called the *instantaneous axis.*

If at any instant the velocities of two points in a rigid body are known, the instantaneous axis will evidently be the line of intersection of the two planes, passing through the points, which are perpendicular to their directions of motion. (See Art. 146.)

In one sense any motion of a rigid body may be considered to be one of rotation, a motion of translation being considered as a motion of rotation about an axis at an infinite distance.

When all the particles of a body move in paths which are parallel to a given plane, the body is said to have *plane motion.* In the following articles we shall consider such cases only.

201. Translation of a Rigid Body.—Let bcd (Fig. 208) be any rigid body which has imparted to it a motion of translation only, its center of gravity O moving in the path AOB, parallel to the plane of the paper. Assume the rectangular coördinate axes OX, OY and OZ, with OX tangent to AOB at O, and OZ perpendicular to the plane of the path of the center of gravity.

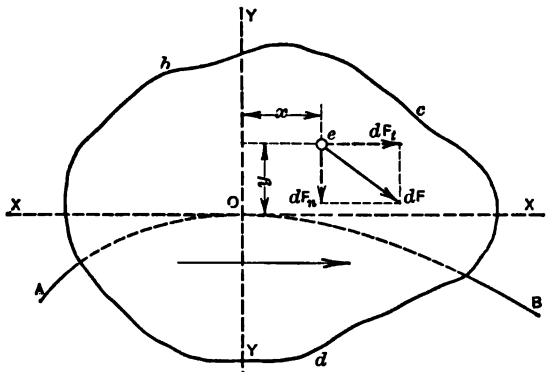

FIG. 208.

The resultant of the external forces acting upon the body at the instant its center of gravity is in the position shown may then be found in the following manner:

Let M = the mass of the body and W = its weight; and let dM = the mass of a particle at any point e, whose coördinates are (x, y, z), v = its velocity and a = its acceleration. The velocity v will be parallel to OX and the acceleration a may be resolved into a tangential component a_t, parallel to OX, and a normal component a_n, parallel to OY (Art. 143).

If we resolve the resultant force dF, acting on dM, into components tangent and normal to its path, the tangential component will be parallel to OX and equal to

$$dF_t = a_t\,dM,$$

and the normal component will be parallel to OY and equal to

$$dF_n = \frac{v^2}{r}dM \text{ (Art. 150)},$$

where r = the radius of curvature of AOB at O.

If we resolve in the same manner the resultant forces acting on all the particles, we shall have a system of forces composed of the tangential forces, which are all parallel to OX, and the normal, or deviating forces, which are all parallel to OY. The algebraic sum of the components parallel to OX will be equal to

$$\Sigma X = a_t \int dM = Ma_t, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and of the components parallel to OY,

$$\Sigma Y = \frac{v^2}{r}\int dM = \frac{Mv^2}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence, if we resolve the resultant of the system of external forces acting on the body into components F_t, parallel to OX, and F_n, parallel to OY, we shall have (Art. 199)

$$F_t = \Sigma X = Ma_t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$F_n = \Sigma Y = \frac{Mv^2}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The lines of action of F_t and F_n may be determined as follows:

The moment of the component dF_t about OY will be equal to

$$z\,dF_t = a_t z\,dM,$$

and about OZ,

$$y\,dF_t = a_t y\,dM.$$

The moment of the component dF_n about OX will be equal to

$$z\,dF_n = \frac{v^2}{r} z\,dM,$$

and about OZ,

$$x\,dF_n = \frac{v^2}{r} x\,dM.$$

The sum of the moments about OY of the tangential components acting on all the particles will be equal to

$$\Sigma M_{y_t} = a_t \int z\,dM,$$

and about OZ,

$$\Sigma M_{z_t} = a_t \int y\,dM.$$

The sum of the moments about OX of the deviating forces acting on all the particles will be equal to

$$\Sigma M_{x_n} = \frac{v^2}{r}\int z\,dM,$$

and about OZ,

$$\Sigma M_{z_n} = \frac{v^2}{r}\int x\,dM.$$

Since the origin is taken at the center of gravity,

$$\int x\,dM = 0, \int y\,dM = 0 \text{ and } \int z\,dM = 0 \text{ (Art. 92)};$$

and hence $\Sigma M_{y_t} = 0$, $\Sigma M_{z_t} = 0$, $\Sigma M_{x_n} = 0$ and $\Sigma M_{z_n} = 0$.

Hence the lines of action of both the components, F_t and F_n, pass through O and the resultant of the system of forces acting on the body will be a single force whose magnitude is equal to

$$R = \sqrt{F_t^2 + F_n^2},$$

and whose line of action passes through the center of gravity of the body.

For convenience we will call the component F_t the *tangential force* and the component F_n the *deviating force*.

When the motion is rectilinear, the deviating force F_n is equal to zero and the resultant force R is equal to the tangential force F_t; and its line of action coincides with the path of the center of gravity.

Therefore the line of action of the resultant of a system of forces, producing a motion of translation only in a rigid body, will pass through its center of gravity; and the magnitudes of the resultant force and its tangential and deviating components will be the same as if the mass of the body were concentrated at its center of gravity.

202. Momentum of Body having a Motion of Translation. — The momentum of any particle e, of mass dM (Fig. 208), will be equal to $v\,dM$ and, since the velocities of all the particles are equal, the sum of their momenta will be equal to

$$v\int dM = Mv, \quad \ldots \ldots \quad (1)$$

that is, the total momentum of the body is the same as if its mass were concentrated at its center of gravity.

Since $$F_t = Ma_t \text{ (Art. 201)},$$

$$F_t\,dt = Ma_t\,dt = M\,dv, \quad \ldots \ldots \quad (2)$$

and hence

$$\int_0^t F_t\,dt = \int_{v_0}^{v_1} M\,dv = M(v_1 - v_0), \quad \ldots \quad (3)$$

where v_0 = the initial velocity and v_1 = the velocity of the body at the end of the time t.

When F_t = a constant, equation (3) becomes

$$F_t t = M(v_1 - v_0). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Since the deviating force

$$F_n = \frac{Mv^2}{r} \text{ (Art. 201)}$$

will produce no change in the speed of the body, its only effect on the momentum will be to change its direction. *Hence the change in momentum produced by a system of forces acting on a body having a motion of translation is the same as if its mass were concentrated at its center of gravity.*

203. Kinetic Energy of a Body having a Motion of Translation. — The kinetic energy of any particle e, of the mass M (Fig. 208), will be equal to

$$\frac{v^2}{2} dM,$$

and, since the velocities of all the particles are equal, the kinetic energy of the entire mass will be equal to

$$E = \frac{v^2}{2}\int dM = M\frac{v^2}{2} = \frac{Wv^2}{2g}. \quad . \quad . \quad . \quad . \quad (1)$$

This quantity may be called the *energy of translation.*

Equation (2) (Art. 202) may be written

$$F_t v\, dt = Mv\, dv; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

but $v\,dt = ds$, the space passed over during the time dt, and hence

$$\int_0^s F_t\, ds = M\int_{v_0}^{v_1} v\, dv = \frac{M}{2}(v_1^2 - v_0^2), \quad . \quad . \quad . \quad (3)$$

where v_0 = the initial velocity and v_1 = the velocity of the body after moving through the distance s.

If F_t = a constant equation (3) becomes

$$F_t s = \frac{M}{2}(v_1^2 - v_0^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Since the deviating force F_n (Art. 201) will have no effect on the speed, *the change in kinetic energy produced by a system of forces acting on a body having a motion of translation is the same as if its mass were concentrated at its center of gravity.*

204. Centripetal Force Acting on a Body having a Motion of Translation in a Circular Path. — When the center of gravity of a body having a motion of translation moves with a uniform speed in a circular path, all its particles move with equal speeds in circular paths of equal radii and the resultant deviating force

$$F_n = \frac{Mv^2}{r} \text{ (Art. 201)}$$

may be called the centripetal force.

A further discussion of centripetal and centrifugal forces will be taken up in connection with rotation.

205.* Rotation of Rigid Bodies About Fixed Axes. — Let *bcd* be any rigid body which has imparted to it a motion of rotation, in the direction indicated (Fig. 209), about a fixed axis through O, perpendicular to the plane of the paper.

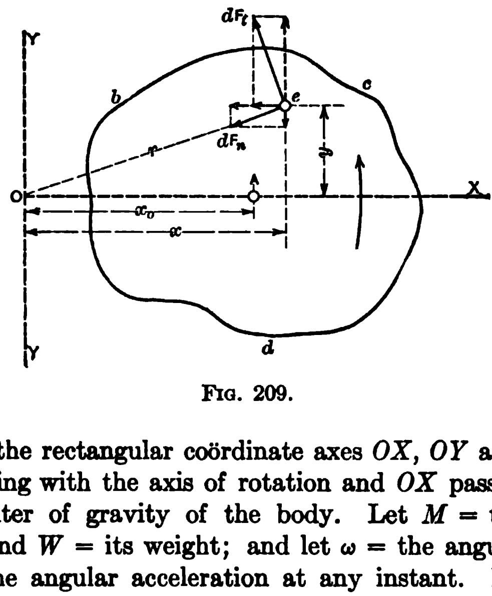

FIG. 209.

Assume the rectangular coördinate axes OX, OY and OZ, with OZ coinciding with the axis of rotation and OX passing through A, the center of gravity of the body. Let M = the mass of the body and W = its weight; and let ω = the angular velocity and α = the angular acceleration at any instant. For convenience the Z plane will be designated as the *plane of rotation.*

The resultant of the external forces acting on the body at the instant its center of gravity is in the position shown, and its components in the directions of the axes OX and OY, may then be found in the following manner:

* See Appendix.

Let dM = the mass of a particle at any point e, whose coördinates are (x, y, z) and whose perpendicular distance from OZ is equal to r. Let v = its linear velocity and a = its linear acceleration.

The direction of v will be perpendicular to Oe; and a may be resolved into a tangential component a_t, perpendicular to Oe and a normal component a_n, along Oe (Art. 143).

If we resolve the resultant force acting on the particle into a tangential component dF_t, and a normal component, or deviating force dF_n, we shall have

$$dF_t = a_t \, dM, \quad \ldots \ldots \quad (1)$$

and

$$dF_n = \frac{v^2}{r} dM \text{ (Art. 150)}. \quad \ldots \ldots \quad (2)$$

If we resolve in the same manner the resultant forces acting on all the particles we shall have a system of forces parallel to the Z plane, which may be divided into two groups: the first group consisting of the tangential forces, and the second group consisting of the deviating forces. At any instant the lines of action of the forces in the first group will coincide with the directions in which the particles are moving, while the forces in the second group will always be perpendicular to the axis of rotation. To avoid confusion we will determine the resultants of the two groups separately.

Tangential Forces.— Substituting $a_t = \alpha r$ (Art. 145) in equation (1) we have

$$dF_t = \alpha r \, dM.$$

Resolving this force into components parallel to OX and OY we obtain for the X component

$$\frac{y}{r} dF_t = \alpha y \, dM,$$

and for the Y component

$$\frac{x}{r} dF_t = \alpha x \, dM.$$

Resolving in a similar manner the tangential forces acting on all the particles into components parallel to the coördinate axes and finding their algebraic sums, we obtain

$$\Sigma X_t = \alpha \int y \, dM = \alpha y_0 M \text{ (Art. 92)}, \quad \ldots \ldots \quad (3)$$

$$\Sigma Y_t = \alpha \int x\, dM = \alpha x_0 M, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\Sigma Z_t = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Since the Y plane passes through the center of gravity, $y_0 = 0$, and hence, if we let

$$F_t = \sqrt{(\Sigma X_t)^2 + (\Sigma Y_t)^2 + (\Sigma Z_t)^2} \text{ (Art. 71)},$$

we shall have

$$F_t = \alpha x_0 M = M a_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where a_0 = the tangential component of the linear acceleration of the center of gravity of the body.

The moment of dF_t about OZ will be equal to

$$r\, dF_t = \alpha r^2\, dM,$$

about OX,

$$z \frac{x}{r} dF_t = \alpha zx\, dM.$$

and about OY,

$$z \frac{y}{r} dF_t = \alpha zy\, dM.$$

The sums of the moments about the axes OZ, OX and OY of the tangential components acting on all the particles will be respectively equal to

$$M_0 = \Sigma M_{z_t} = \alpha \int r^2\, dM = \alpha I_m \text{ (Art. 129)}, \quad . \quad . \quad . \quad (7)$$

$$\Sigma M_{x_t} = \alpha \int zx\, dM = \alpha K_{xz_m} \text{ (Art. 138)} \quad . \quad . \quad (8)$$

and

$$\Sigma M_{y_t} = \alpha \int zy\, dM = \alpha K_{yz_m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The resultant of the three couples (equations 7, 8 and 9) might be found (Art. 71), but it will be simpler to determine the effect of each one separately. This will be done after determining the resultant of the deviating forces acting on the particles.

Deviating Forces.—Substituting $v = \omega r$ (Art. 144) in equation (2) we have

$$dF_n = \omega^2 r\, dM.$$

Resolving this force into components parallel to OX and OY we shall have for the X component

$$\frac{x}{r} dF_n = \omega^2 x\, dM,$$

and for the Y component

$$\frac{y}{r}\,dF_n = \omega^2 y\,dM.$$

Resolving in a similar manner the deviating forces acting on all the particles into components parallel to the coördinate axes and finding their algebraic sums, we obtain

$$\Sigma X_n = \omega^2 \int x\,dM = \omega^2 x_0 M, \quad \ldots \quad (10)$$

$$\Sigma Y_n = \omega^2 \int y\,dM = \omega^2 y_0 M, \quad \ldots \quad (11)$$

$$\Sigma Z_n = 0. \quad \ldots \quad (12)$$

Since the Y plane passes through the center of gravity, $y_0 = 0$, and hence, if we let

$$F_n = \sqrt{(\Sigma X_n)^2 + (\Sigma Y_n)^2 + (\Sigma Z_n)^2} \text{ (Art. 71)},$$

we shall have

$$F_n = \omega^2 x_0 M = M\frac{v_0^2}{x_0}, \quad \ldots \quad (13)$$

where $v_0 =$ the linear velocity of the center of gravity of the body.

The moment of dF_n about OX will be equal to

$$z\frac{y}{r}\,dF_n = \omega^2 zy\,dM,$$

and about OY,

$$z\frac{x}{r}\,dF_n = \omega^2 zx\,dM.$$

The sums of the moments about the axes OZ, OX and OY of the deviating forces acting on all the particles will be respectively equal to

$$\Sigma M_{z_n} = 0, \quad \ldots \quad (14)$$

$$\Sigma M_{x_n} = \omega^2 \int zy\,dM = \omega^2 K_{yz_m}, \quad \ldots \quad (15)$$

$$\Sigma M_{y_n} = \omega^2 \int zx\,dM = \omega^2 K_{xz_m}. \quad \ldots \quad (16)$$

The resultant of these three couples (equations 14, 15 and 16) might be found (Art. 71) but it will be simpler to determine the effect of each one separately.

It is evident from the preceding that in general the resultant of the tangential forces acting on the particles of a rotating body

will be a single force, whose magnitude (equation 6) is the same as if the mass of the body were concentrated at its center of gravity, and three component couples whose magnitudes (equations 7, 8 and 9) will depend on the moment of inertia of the mass about OZ and the products of inertia of the mass with respect to the X and Z planes and the Y and Z planes, respectively.

It is also evident that the resultant of the deviating forces acting on the particles of a rotating body will be a single force, whose magnitude (equation 13) is the same as if the mass of the body were concentrated at its center of gravity, and two component couples whose magnitudes (equations 15 and 16) depend upon the values of the products of inertia of the mass with respect to the Y and Z planes, and the X and Z planes, respectively. Special cases will arise when one, or both, of the component forces and part, or all, of the component couples are equal to zero. These will be considered under the following four cases:

CASE I. — When $K_{xz_m} = 0$ and $K_{yz_m} = 0$.

In this case the couples ΣM_{x_t}, ΣM_{y_t}, ΣM_{x_n} and ΣM_{y_n} (equations 8, 9, 15 and 16) will all be equal to zero.

Hence the resultant of the *tangential forces* will be the force

$$F_t = \alpha x_0 M \text{ (equation 6)}, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and its line of action will be in the Z plane, parallel to the axis OY, and at a perpendicular distance from OZ which will be equal to

$$l = \frac{\Sigma M_{z_t}}{F_t} = \frac{\alpha I_m}{\alpha x_0 M} = \frac{I_m}{x_0 M} = \frac{\rho^2}{x_0}, \quad . \quad . \quad . \quad . \quad (18)$$

where ρ = the radius of gyration of the mass about OZ.

The resultant of the *deviating forces* will be the force

$$F_n = \omega^2 x_0 M \text{ (equation 13)}, \quad . \quad . \quad . \quad . \quad (19)$$

and its line of action will coincide with the axis OX and intersect the line of action of F_t at a point whose distance from O is equal to l.

Therefore the resultant of the system of external forces acting on the body will be a single force whose components are F_t and F_n (Art. 199), whose magnitude will be equal to

$$R = \sqrt{(F_t)^2 + (F_n)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and whose line of action will be in the Z plane and make an angle with OX equal to

$$\sin^{-1}\frac{F_t}{R}.$$

The moment of the resultant force about the axis of rotation will evidently be equal to

$$M_0 = F_t l = \alpha I_m. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The point of intersection of F_t and F_n through which the line of action of the resultant of all the external forces acting on the body will pass, is called the *center of percussion* with respect to the axis of rotation OZ.

A common example under this case, occurring in engineering problems, is that of a body which is symmetrical with respect to the plane of rotation of its center of gravity. The body rotating about the axis OZ, which is perpendicular to a plane of symmetry through its center of gravity A (Fig. 210), may be taken as an illustration. Referring the body to the rectangular

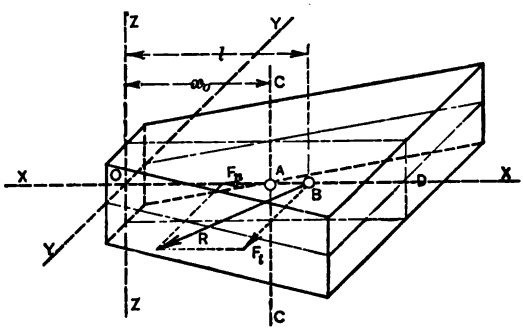

FIG. 210.

axes OX, OY and OZ, with OX passing through A, the products of inertia K_{zx_m} and K_{zy_m} will each be equal to zero (Art. 139); and the lines of action of the accelerating and deviating forces F_t and F_n will pass through the center of percussion B and lie in the Z plane.

If the motion is uniform, $F_t = 0$, and the resultant force acting on the body will be the deviating force F_n.

A straight rod of uniform, or varying, cross section symmetrical with respect to the plane of rotation, which rotates about an axis perpendicular to its length, and the ordinary forms of the com-

pound pendulum are examples of rotating bodies coming under this case.

If the body rotates about the axis CC, parallel to OZ, through its center of gravity we shall have a special case, which may be designated as:

Case Ia. — In this case $x_0 = 0$, and we shall have, in addition to the conditions already enumerated under Case I, $F_t = 0$ and $F_n = 0$ (equations 6 and 13). Hence the resultant of the external forces acting on the body will be a couple, parallel to the Z plane, whose moment is equal to

$$M_0 = \alpha I_m \text{ (equation 7)}. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

This may be called the *accelerating couple.*

When the motion is uniform, $\alpha = 0$; and hence $M_0 = 0$ and the system of forces acting on the body is in equilibrium.

The ordinary flywheel and pulley are examples of bodies coming under this case. A homogeneous body rotating about an axis OZ through its center of gravity and symmetrical with respect to the X and Y planes will come under this case, even if it is unsymmetrical with respect to the Z plane, since for such a body $K_{xz_m} = 0$ and $K_{yz_m} = 0$ (Art. 139).

Although it is possible that the conditions under Case I may be fulfilled in the case of non-symmetrical and non-homogeneous bodies, such cases do not ordinarily occur in problems where accurate computations are required.

Case II. — When $K_{xz_m} >< 0$ and $K_{yz_m} = 0$.

In this case $\Sigma M_{y_t} = 0$ and $\Sigma M_{z_n} = 0$ (equations 9 and 15).

Hence the resultant of the *tangential forces* will be the force

$$F_t = \alpha x_0 M \text{ (equation 6)}, \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and its line of action will be parallel to the axis OY, at a perpendicular distance from OZ which is equal to

$$l = \frac{I_m}{x_0 M} \text{ (equation 18)}, \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and at a perpendicular distance from OX which is equal to

$$z_t = \frac{\Sigma M_{x_t}}{F_t} = \frac{\alpha K_{xz_m}}{\alpha x_0 M} = \frac{K_{xz_m}}{x_0 M}. \quad . \quad . \quad . \quad . \quad (25)$$

The resultant of the *deviating forces* will be the force

$$F_n = \omega^2 x_0 M \text{ (equation 13)}, \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and its line of action will be in the Y plane, parallel to the axis OX, and at a perpendicular distance from OY which is equal to

$$z_n = \frac{\Sigma M_{y_n}}{F_n} = \frac{\omega^2 K_{xz_m}}{\omega^2 x_0 M} = \frac{K_{xz_m}}{x_0 M}. \quad . \quad . \quad . \quad . \quad (27)$$

Hence $z_n = z_t$ and the forces F_t and F_n will act through the same point, which is called the *center of percussion* with respect to the axis of rotation OZ.

As in Case I the magnitude of the resultant of the external forces acting on the body will be equal to

$$R = \sqrt{(F_t)^2 + (F_n)^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

its line of action will be parallel to the Z plane and make an angle with the Y plane which is equal to

$$\sin^{-1}\frac{F_t}{R};$$

and the moment of the resultant force about the axis of rotation will be equal to

$$M_0 = F_t l = \alpha I_m. \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The essential difference between Cases I and II is in the location of the line of action of the resultant force R. While in the former it is located in the plane of rotation of the center of gravity of the body, in the latter case it is in a parallel plane, whose perpendicular distance from the Z plane is equal to

$$\frac{K_{xz_m}}{x_0 M}.$$

An example under this case is that of a body, which is symmetrical with respect to a plane containing its center of gravity and the axis of rotation, but not with respect to the plane of rotation. The body rotating about the axis OZ and symmetrical with respect to the Y plane, containing its center of gravity A (Fig. 211), may be taken as an illustration. The product of inertia $K_{yz_m} = 0$ (Art. 139), and the lines of action of the accelerating and deviating forces F_t and F_n will pass through the center of percussion B and be parallel to the Z plane.

If the motion is uniform, $F_t = 0$ and the resultant force acting on the body will be the deviating force F_n.

The rods and weights of the ordinary pendulum governor are examples of rotating bodies of this kind.

If the body rotates about an axis through its center of gravity A, parallel to OZ, we shall have a special case which may be designated as:

Case IIa. — In this case $x_0 = 0$ and we shall have, in addition to the conditions already enumerated under Case II, $F_t = 0$ and $F_n = 0$ (equations 6 and 13). Hence the resultant of the external forces acting on the body will be the system composed of three component couples, viz.:

$$M_0 = \alpha I_m \text{ (equation 7)}, \quad \ldots \quad (30)$$

$$\Sigma M_{x_t} = \alpha K_{xz_m} \text{ (equation 8)} \quad \ldots \quad (31)$$

and

$$\Sigma M_{y_n} = \omega^2 K_{xz_m} \text{ (equation 16)}. \quad \ldots \quad (32)$$

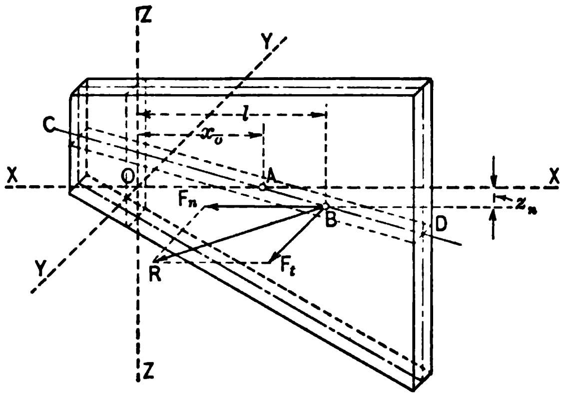

Fig. 211.

The couple M_0 (equation 30), acting parallel to the Z plane, may be called the *accelerating couple.*

The couples ΣM_{x_t} and ΣMy_n (equations 31 and 32) will prevent the displacement of the axis of rotation in the X and Y planes respectively. They will be components of the system of forces acting on the axis, simply, and will have no effect on the speed of rotation. Hence they may be called *balancing couples.*

When the motion is uniform, $\alpha = 0$, and hence $M_0 = 0$ and $\Sigma M_{x_t} = 0$; and the resultant of the system of forces acting on the body will be the couple ΣM_{y_n} (equation 32) acting in the Y plane as a balancing couple to prevent the displacement of the axis of rotation.

A straight rod of uniform or non-uniform section, rotating about a fixed axis, in a plane of symmetry through its center of gravity, but which is not at right angles to the axis of the rod, is an example of a body coming under this case.

As in Case I, the conditions under Case II may be fulfilled by non-symmetrical bodies, but such do not ordinarily occur in problems where accurate computations are to be made.

CASE III. — When $K_{zx_m} = 0$ and $K_{yz_m} >< 0$.

In this case $\Sigma M_{x_t} = 0$ and $\Sigma M_{y_n} = 0$ (equations 8 and 16).

Hence the resultant of the *tangential forces* will be the force F_t (equation 6) acting in the Z plane, parallel to OY and at a perpendicular distance from OZ equal to l (equation 18), together with a couple ΣM_{y_t} (equation 9) in the Y plane.

The resultant of the *deviating* forces will be the force F_n (equation 13) acting along OX, together with a couple ΣM_{z_n} (equation 15) in the X plane.

Hence the resultant of the forces F_t and F_n will act through the center of percussion, which will be located on the axis OX at a distance from OZ equal to l, as in Case I; and the couples ΣM_{y_t} and ΣM_{z_n} will act as balancing couples to prevent the displacement of the axis of rotation in the Y and X planes respectively.

If the motion is uniform, $\alpha = 0$, and hence $F_t = 0$ and $\Sigma M_{y_t} = 0$; and the resultant of the external forces will be the deviating force F_n and the couple ΣM_{z_n}, preventing the displacement of the axis of rotation in the X plane.

When the axis of rotation passes through the center of gravity we shall have a special case which may be designated as:

Case IIIa. — In this case $x_0 = 0$, and we shall have, in addition to the conditions already enumerated under Case III, $F_t = 0$ and $F_n = 0$ (equations 6 and 13), and the resultant of the external forces acting on the body will be the system comprising the three component couples, viz.: the accelerating couple M_0 (equation 7) and the balancing couples ΣM_{y_t} (equation 9) and ΣM_{z_n} (equation 15).

When the motion is uniform, $\alpha = 0$, and hence $M_0 = 0$ and $\Sigma M_{y_t} = 0$; and the resultant of the external forces will be the balancing couple ΣM_{z_n} (equation 15).

It should be noted that Case III differs from Case I only in the condition that a balancing couple in a plane containing the axis of rotation is required to prevent the displacement of that axis,

the formulas for the accelerating and deviating forces being the same as in Case I. A straight rod whose center of gravity is on the axis of X and which neither intersects nor is parallel to the axis of rotation, OZ, is an example under this case.

CASE IV. — When $K_{xz_m} >< 0$ and $K_{yz_m} >< 0$.

In this case the resultant of the *tangential forces* will be the force F_t (equation 6), whose line of action is parallel to OY, at a perpendicular distance from OZ which is equal to l (equation 18), and at a perpendicular distance from OX which is equal to z_t (equation 25), together with a couple, parallel to the Y plane, whose moment is equal to ΣM_{y_t} (equation 9).

The resultant of the *deviating forces* will be the force F_n (equation 13), whose line of action is in the Y plane, parallel to OX, and at a perpendicular distance from OY which is equal to z_n (equation 27), together with a couple, parallel to the X plane, whose moment is equal to ΣM_{x_n} (equation 15).

Hence the resultant of the forces F_t and F_n will act through the center of percussion, which will be located in the Y plane at a distance from OZ equal to l and at a distance from OX equal to $z_t = z_n$, as in Case II; and the couples ΣM_{y_t} and ΣM_{x_n} will act as balancing couples to prevent the displacement of the axis of rotation in the Y and X planes, respectively.

When the motion is uniform, $F_t = 0$ and $\Sigma M_{y_t} = 0$; and the resultant of the external forces will be the force F_n and the balancing couple ΣM_{x_n}, preventing the displacement of the axis of rotation in the X plane.

When the axis of rotation passes through the center of gravity we shall have a special case which may be designated as:

Case IVa. — In this case $x_0 = 0$, and we shall have, in addition to the conditions already enumerated under Case IV, $F_t = 0$ and $F_n = 0$, and the resultant of the external forces acting on the body will be a system comprising the three component couples, viz.: the accelerating couple M_0 (equation 7) and the balancing couples $M_x = \Sigma M_{x_t} + \Sigma M_{x_n}$ (equations 8 and 15) and $M_y = \Sigma M_{y_t} + \Sigma M_{y_n}$ (equations 9 and 16).

When the motion is uniform, $\alpha = 0$, and hence $M_0 = 0$, $\Sigma M_{x_t} = 0$ and $\Sigma M_{y_t} = 0$; and the resultant of the system of external forces acting on the body will be the resultant of the balancing couples ΣM_{x_n} and ΣM_{y_n}.

It should be noted that Case IV differs from Case II only in the condition that a balancing couple in a plane containing the

axis of rotation is required to prevent the displacement of that axis, the formulas for the accelerating and deviating forces being the same as in Case II.

Case IV may be regarded as the general case covering all cases of non-symmetrical bodies rotating about fixed axes.

SUMMARY. — If we refer a body, to which is imparted a motion of rotation about a fixed axis, to three rectangular coördinate axes, with the axis OZ coinciding with the axis of rotation and the axes OX and OY rotating with the body, in such a manner that the axis OX always passes through the center of gravity, the results obtained under the preceding four cases may be summarized as follows:

(a) In every case the sum of the moments, about the axis of rotation, of the external forces acting on the body will be equal to

$$M_0 = \alpha I_m = \frac{\alpha I}{g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

(b) In every case, if the axis of rotation passes through the center of gravity, M_0 = a couple acting in the Z plane.

(c) In every case in which the axis of rotation does not pass through the center of gravity

$$M_0 = F_t l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where $$F_t = \alpha x_0 M = M a_0 = \frac{W}{g} a_0 \quad . \quad . \quad . \quad . \quad . \quad (35)$$

is the resultant tangential force acting through the center of percussion, parallel to OY, and

$$l = \frac{I_m}{x_0 M} = \frac{I}{x_0 W} = \frac{\rho^2}{x_0} = \frac{I_0}{x_0 W} + x_0 = \frac{\rho_0^2}{x_0} + x_0, \quad . \quad (36)$$

where I_0 = the moment of inertia and ρ_0 = the radius of gyration about an axis through the center of gravity parallel to OZ.

(d) In every case in which the axis of rotation does not pass through the center of gravity the resultant of the system of external forces acting on the body will always include, in addition to the force F_t, a resultant deviating force

$$F_n = \omega^2 x_0 M = \frac{\omega^2}{g} W x_0, \quad . \quad . \quad . \quad . \quad . \quad (37)$$

acting through the center of percussion and perpendicular to OZ.

(*e*) When the axis of rotation does not pass through the center of gravity; if the product of inertia $K_{zz} = 0$, the center of percussion is located on the axis OX; and if $K_{zz} >< 0$, the center of percussion is located in the Y plane at a perpendicular distance from OX which is equal to

$$z_t = z_n = \frac{K_{zz_m}}{x_0 M} = \frac{K_{zz}}{x_0 W}. \quad . \quad . \quad . \quad . \quad . \quad (38)$$

(*f*) In every case where the product of inertia $K_{yz} = 0$ and the axis of rotation OZ does not pass through the center of gravity, the resultant of the external forces will be a single force

$$R = \sqrt{(F_t)^2 + (F_n)^2}, \quad . \quad . \quad . \quad . \quad . \quad (39)$$

acting through the center of percussion.

(*g*) In every case where $K_{yz} >< 0$, one or more balancing couples are required to prevent the displacement of the axis of rotation. These couples will act in planes perpendicular to the plane of rotation and will have no effect on the speed of the body.

(*h*) In every case where the axis of the rotation OZ passes through the center of gravity and $K_{yz} >< 0$, or $K_{zz} >< 0$, one or more balancing couples are required to prevent the displacement of the axis of rotation. These couples will have no effect on the speed of the body.

(*i*) When the axis of rotation OZ passes through the center of gravity and the products of inertia $K_{zz} = 0$ and $K_{yz} = 0$, no balancing couples are required to prevent a displacement of the axis. In this case OZ is a principal axis (Art. 140) of the rotating body, and, if it is the axis about which the moment of inertia is a maximum, its position may be said to be stable: that is, the body will tend to continue to rotate in the same plane if any slight displacement of the axis occurs; whereas, if it is one of the other principal axes, force will be required to prevent a change in the plane of rotation, if any slight displacement of the axis occurs.

206. Angular Momentum. — Following the notation in Art. 205, the momentum at any instant of the particle at e (Fig. 209) will be equal to

$$v\,dM = \omega r\,dM,$$

and the moment of its momentum (Art. 149) about the axis OZ will be equal to

$$\omega r^2\,dM.$$

The sum of the moments for all the particles in the body will be equal to

$$\omega \int r^2\, dM = \omega I_m = \omega \frac{I}{g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The quantity $\omega \frac{I}{g} = \omega I_m$ is called the *angular momentum*, or the *moment of the momentum* of the body.

If we substitute in equation (33) (Art. 205) the value for the angular acceleration

$$\alpha = \frac{d\omega}{dt},$$

we shall have

$$M_0 = I_m \frac{d\omega}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Integrating this expression we shall obtain

$$\int_0^t M_0\, dt = I_m \int_{\omega_0}^{\omega_1} d\omega = I_m (\omega_1 - \omega_0), \quad . \quad . \quad . \quad . \quad (3)$$

which is the expression for the change in angular momentum produced by a system of forces whose moment about the axis of rotation at any instant is equal to M_0, ω_0 being the initial angular velocity and ω_1 the angular velocity at the end of the time t. If M_0 is a constant, equation (3) becomes

$$M_0 t = I_m (\omega_1 - \omega_0) = \frac{I}{g} (\omega_1 - \omega_0). \quad . \quad . \quad . \quad . \quad (4)$$

207. Kinetic Energy of a Rotating Body. — Following the notation in Art. 205, the kinetic energy at any instant of the particle at e (Fig. 209) will be equal to

$$\frac{v^2}{2} dM = \frac{\omega^2 r^2}{2} dM.$$

The sum of the energies of all the particles in the body will be equal to

$$E_r = \frac{\omega^2}{2} \int r^2\, dM = \frac{\omega^2}{2} I_m = \frac{\omega^2 I}{2g}. \quad . \quad . \quad . \quad . \quad (1)$$

This quantity may be called the *energy of rotation*. Since $I = \rho^2 W$ (Art. 131),

$$E_r = \frac{\omega^2 \rho^2 W}{2g} = \frac{v_\rho^2 W}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where v_ρ is the linear velocity of a point at the distance ρ from the axis. Hence the kinetic energy of a rotating body is the same as if its mass were concentrated at a point whose distance from the axis of rotation is equal to its radius of gyration. Equation (2) (Art. 206) may be written

$$M_0\omega\, dt = I_m\omega\, d\omega.$$

But $\omega\, dt = d\theta$, the angular displacement of the body during the time dt. Hence by integration

$$\int_0^\theta M_0\, d\theta = I_m \int_{\omega_0}^{\omega_1} \omega\, d\omega = \frac{I_m}{2}(\omega_1^2 - \omega_0^2), \quad . \quad . \quad . \quad (3)$$

where ω_0 and ω_1 are the initial and final angular velocities and $\theta =$ the total angle through which the body turns during the change in velocity.

If M_0 is a constant, equation (3) becomes

$$M_0\theta = \frac{I_m}{2}(\omega_1^2 - \omega_0^2) = \frac{I}{2g}(\omega_1^2 - \omega_0^2). \quad . \quad . \quad . \quad (4)$$

It is evident that

$$M_0\theta = M_0 2\pi N, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $N =$ the number of revolutions through which the body turns during the change in velocity.

208. Centripetal and Centrifugal Forces.—When a body rotates about a fixed axis with a uniform angular velocity, the formula

$$F_n = \omega^2 x_0 M \text{ (equation 37, Art. 205)}$$

will give the magnitude of the resultant centripetal force acting on the body.

The centrifugal force will be the reaction, equal and opposite to the centripetal force, exerted on the axis of rotation. The line of action of this force will always pass through the center of percussion and be perpendicular to the axis of rotation.

In almost all engineering problems involving rotation about a fixed axis, it is customary to refer to the effect of the centrifugal force rather than to the centripetal force, which is the force actually exerted on the rotating body; and sometimes the term is used rather loosely. For example, we use such expressions as: the pull on the axis due to the centrifugal force of a revolving weight; the tension in a rotating ring due to centrifugal force; the effect of

centrifugal force when a car goes around a curve, the centrifugal action of the parts of a rotating body, and similar expressions; where the centrifugal force is the reaction at the axis of rotation due to the centripetal force, acting on the body, to produce the change in the direction of its motion.

209. Interchangeability of the Center of Percussion and the Center of Rotation. — The center of percussion has been defined (Art. 205) as the point of application of the resultant force, of a system of forces producing a motion of rotation in a body about an axis.

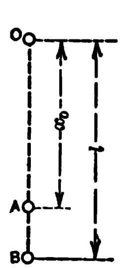

FIG. 212.

Another definition is the following: The center of percussion of a body is the point at which a force may be applied suddenly without producing a shock on the axis of rotation.

Proposition. — If the point O (Fig. 212) is the projection of the axis of rotation of a body, A the projection of its center of gravity and B the projection of its center of percussion on the plane of rotation; then, if the body is made to rotate about a parallel axis through B, will O be the projection of the new center of percussion.

Proof. — Let $OA = x_0$, and $OB = l$.
When the axis of rotation is through O,

$$l = \frac{\rho^2}{x_0} \text{ (Art. 205)}, \quad \ldots \ldots \ldots \quad (1)$$

and since

$$\rho^2 = \rho_0^2 + x_0^2 \text{ (Art. 133)},$$

$$l = \frac{\rho_0^2}{x_0} + x_0. \quad \ldots \ldots \ldots \quad (2)$$

If the body is made to rotate about an axis through B, parallel to the axis through O, we shall have

$$l_1 = \frac{\rho_1^2}{x_{0_1}}, \quad \ldots \ldots \ldots \quad (3)$$

where $\rho_1 =$ the radius of gyration about the axis through B, and $x_{0_1} = AB$, the distance of the center of gravity and $l_1 =$ the distance of the new center of percussion from that axis.

But
$$x_{0_1} = l - x_0,$$
and
$$\rho_1^2 = \rho_0^2 + x_{0_1}^2 = \rho_0^2 + (l - x_0)^2,$$

and hence

$$l_1 = \frac{\rho_0^2}{x_{0_1}} + x_{0_1} = \frac{\rho_0^2}{l - x_0} + l - x_0 \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

From equation (2) we obtain

$$x_0 = \frac{\rho_0^2}{l - x_0},$$

and substituting this value in equation (4) we have

$$l_1 = x_0 + l - x_0 = l. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5) \text{ Q. E. D.}$$

210. Center of Oscillation. — The center of oscillation of a compound pendulum is the point at which, if the whole mass were concentrated, the time of oscillation would remain unchanged. Its distance from the axis of oscillation is the length of the equivalent simple pendulum.

Proposition — The center of oscillation of a compound pendulum coincides with its center of percussion.

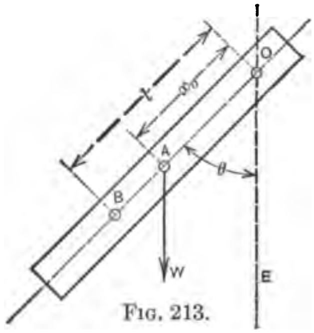

FIG. 213.

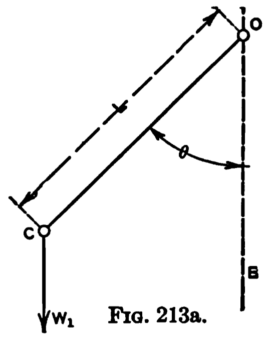

FIG. 213a.

Proof. — Let O be the axis of oscillation, A the center of gravity, B the center of percussion and W the weight of a compound pendulum (Fig. 213); and let $x_0 = OA$ and $l = OB$. Let L = the length OC, of the equivalent simple pendulum (Fig. 213a), and W_1 = its weight.

Since the pendulums will oscillate in the same time, the angular accelerations of the two, when their center lines make a given angle θ with the verticals through the axes of oscillation, will be the same.

The forces acting on the compound pendulum will be the force of gravity, acting vertically downward through A, and the force exerted by the axis through O, whose direction at any instant will depend on the angular velocity at that time. If the pendulum is not symmetrical with respect to the plane of rotation through A,

there may be a couple exerted on the axis O in addition to the above mentioned force. This couple would be in a plane perpendicular to the plane of rotation and would have no effect on the speed of the pendulum.

The forces acting on the simple pendulum will be the force of gravity acting through C and the pull on the weightless cord OC at O. Let α = the angular acceleration of each pendulum for the position shown.

For the compound pendulum the sum of the moments of the forces about the axis of rotation will be equal to

$$M_0 = Wx_0 \sin\theta = \frac{\alpha I}{g},$$

and

$$\alpha = \frac{gWx_0 \sin\theta}{I} = \frac{g\sin\theta}{l} \text{ (Art. 205)}. \quad . \quad . \quad . \quad (1)$$

For the simple pendulum

$$M_0 = W_1L\sin\theta = \frac{\alpha I}{g} = \frac{\alpha W_1L^2}{g},$$

and

$$\alpha = \frac{g\sin\theta}{L}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equating the values of α in equations (1) and (2) we obtain

$$\frac{g\sin\theta}{l} = \frac{g\sin\theta}{L},$$

and hence

$$l = L. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3) \text{ Q.E.D.}$$

211. Rotation and Translation Combined — Instantaneous Axes. In the preceding articles we have determined the resultant of the system of forces required to impart to a rigid body a plane motion of translation (Art. 201); and also the resultant of a system of forces required to impart a motion of rotation about a fixed axis (Art. 205).

The third case mentioned in Art. 200 will now be considered; namely, that in which a rigid body has imparted to it a plane motion which may be defined as a motion of rotation about an axis which is constantly changing in such a manner that its successive positions are always parallel, the position of the axis of rotation at any instant being called the *instantaneous axis*.

In such a case the motion of the body may be resolved in the following manner: Let A be any point in the body, situated in the

plane of motion of its center of gravity, and B any other point in that plane. The motion of B relative to A, will be one of rotation about A (Art. 146); and the resultant velocity v, of B relative to any fixed point O in the plane of motion, will be equal to the vector sum of the component v_b, due to the motion of rotation relative to A, and the component v_a, due to the motion of A relative to the fixed point (Fig. 157b).

Since A and B are any two points in the plane of motion of a rigid body, it follows that the motion of the body at any instant may be resolved into a motion of rotation, relative to any axis perpendicular to the plane of motion at any point A, and a motion of translation with a velocity equal to that of the point A. The instantaneous axis will pass through the point of intersection O, of the perpendiculars OA and OB to the vectors v_a and v, representing the resultant velocities of the points A and B, respectively. (Fig. 214.)

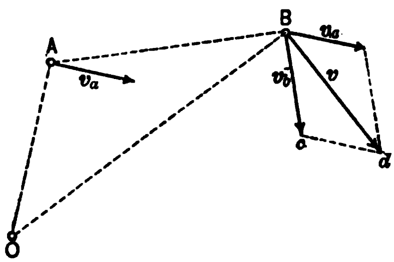

Fig. 214.

Moreover, if ω = the angular velocity of the body about the instantaneous axis O, we shall obtain from the similar triangles OAB and dcB

$$\omega = \frac{v_a}{OA} = \frac{v}{OB} = \frac{v_b}{AB}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Therefore, the angular velocity of B at any instant, and hence of the entire body, relative to the axis through A, is equal to the the angular velocity of the body about the instantaneous axis O.

It is evident that the foregoing will apply when the point O is fixed, that is, for a body rotating about a fixed axis, as well as when the axis of rotation is changing. Hence we may state the following: *When a body has imparted to it a plane motion of rotation about a moving or a fixed axis, the motion may be resolved into a*

motion of rotation relative to an axis perpendicular to the plane of motion through any point A, with an angular velocity ω equal to that of the body about the axis of rotation, and a motion of translation with a linear velocity v_a equal to that of the point A.

When the motion of a rigid body is resolved into rotation and translation in the above manner, for the sake of brevity we shall call ω the velocity of rotation and v_a the velocity of translation.

The acceleration of a body having the motion described above may be treated in the following manner: If a_b = the acceleration of B, relative to A, and a_a = the acceleration of A, relative to any fixed point, the resultant acceleration a, of the point B with respect to this fixed point, will be equal to the vector sum of the accelerations a_a and a_b (Art. 147).

The component acceleration a_b will be composed of a tangential component a_{b_t} and a normal component a_{b_n} (Fig. 158b) and, since the path of the point B relative to A is circular, we may write

$$a_{b_t} = \alpha r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$a_{b_n} = \omega^2 r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where α = the angular acceleration and ω = the angular velocity of rotation about the axis through A and $r = AB$.

If the motion of the point A is rectilinear, the component acceleration a_a will be in the direction of the motion. If the motion of A is curvilinear, a_a will be composed of a tangential and a normal component, the magnitudes of which will depend on the form of the path and the velocity v_a of the point A (Art. 143).

Following the method of designating velocities, when the motion of the body is resolved in the foregoing manner we shall call α the acceleration of rotation and a_a the acceleration of translation.

The resultant of the system of forces, required to impart to a rigid body the motion described above, may be determined by finding the resultant of the system of forces required to impart the motion of rotation (Art. 205) relative to the axis A, and the resultant of the system of forces required to impart the motion of translation (Art. 201), and combining the two resultants.

While the point A may be any point within the rotating body, in most problems dealing with motion of this kind the simplest solution can be made by taking A at the center of gravity. We will, therefore, take up this case first.

CASE I. — *Where A is the center of gravity of the body.*

Let *abc* be a rigid body, having at a given instant a plane motion of rotation with an angular velocity ω about the instantaneous axis through O, perpendicular to the plane XOY (Fig. 215). Let A be the center of gravity of the body, W = its weight, I_0 = its moment of inertia about an axis through A, parallel to the instantaneous axis, and $x_0 = OA$, the perpendicular distance between the axes.

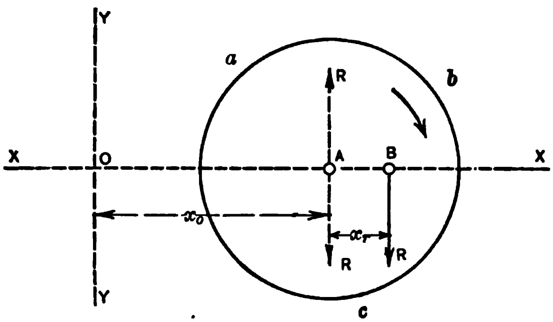

FIG. 215.

The motion may be resolved into a motion of rotation about the axis through A, with an angular velocity ω, and a motion of translation with a linear velocity

$$v_a = \omega x_0, \qquad (4)$$

equal to that of the point A.

The acceleration of rotation will be equal to

$$\alpha = \frac{d\omega}{dt}, \qquad (5)$$

and, since A is the center of gravity, the resultant of the forces required to impart the rotation about the axis through A will be equal to the couple

$$M_0 = \alpha I_m = \frac{\alpha I_0}{g} \text{ (Art. 205).} \qquad (6)$$

As in the case of rotation about a fixed axis, the product of inertia of the body with respect to the plane of motion and one at right angles must be equal to zero, otherwise balancing couples would be required to prevent a change in direction of the axis of rotation.

In discussing the motion of translation two cases will arise.

(*a*) *When the motion of A is rectilinear.* — In this case the acceleration of translation will be equal to

$$a_a = \frac{dv_a}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and the resultant of the forces required to impart the translation will be a force

$$R = \frac{W}{g} a_a \text{ (Art. 201)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

acting through A in the direction of the motion.

The resultant of the entire system of forces acting on the body will, therefore, be the resultant of the force R and the couple M_0, which will be a single force equal and parallel to R, acting in a plane whose perpendicular distance from A is equal to

$$x_r = \frac{M_0}{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If x_0, the distance between the instantaneous axis and the center of gravity, is constant we obtain from equations (7) (4) and (5)

$$a_a = \frac{dv_a}{dt} = x_0 \frac{d\omega}{dt} = \alpha x_0 \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and equation (8) may be written

$$R = \frac{Wa_a}{g} = \frac{W}{g} \alpha x_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Substituting in equation (9) we have

$$x_r = \frac{M_0}{R} = \frac{\alpha I_0}{g} \times \frac{g}{\alpha W x_0} = \frac{I_0}{W x_0} = \frac{\rho_0^2}{x_0} = l - x_0 \text{ (Art. 209)}. \quad . \quad . \quad (12)$$

Therefore, when x_0 = a constant, B, the center of percussion of the body with respect to the instantaneous axis O, is the same point as if O were a fixed axis.

A wheel rolling without slipping along a straight track is a simple illustration under this case. The instantaneous axis is always at the point of contact of the wheel and the track, and the motion may be resolved into a motion of rotation about the axis of the wheel and a rectilinear motion of translation which is the same as that of the center.

(*b*) *When the motion of A is curvilinear.* — In this case the

acceleration of translation may be resolved into the tangential component

$$a_{a_t} = \frac{dv_a}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and the normal component

$$a_{a_n} = \frac{v_a^2}{r_a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where r_a is the radius of curvature of the path of A at the given instant. Hence the resultant of the forces required to impart the translation (Art. 201) will consist of a tangential component,

$$F_t = \frac{W}{g} a_{a_t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and a normal component,

$$F_n = \frac{W}{g} a_{a_n} = \frac{W v_a^2}{g r_a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The resultant R of the entire system of forces acting on the body will, therefore, be the resultant of the couple M_0 and the forces F_t and F_n which will be the same as the resultant of a force equal and parallel to F_t, acting in a plane whose perpendicular distance from A, is equal to

$$x_r = \frac{M_0}{F_t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and the force F_n, acting through A towards the center of curvature of its path.

Hence $$R = \sqrt{F_t^2 + F_n^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

As in the preceding case, if x_0 = a constant,

$$a_{a_t} = \alpha x_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and the center of percussion B will be the same point as if O were a fixed axis.

If we assume that A (Fig. 215) moves in a path whose center of curvature is on OX, the force F_n may be represented by a vector along OA and the force F_t by a vector perpendicular to OA at the point A. The resultant R would then be represented by the diagonal of a rectangle constructed at B with the vectors F_t and F_n as sides.

A wheel rolling without slipping along a curved surface is an

illustration of the motion described above. The instantaneous axis is always at the point of contact of the wheel and surface, and the motion may be resolved into a motion of rotation about the axis of the wheel and a motion of translation along the curve followed by the center of the wheel.

CASE II. — *Where A is not the center of gravity of the body.*

We shall consider only the case when the motion of A is rectilinear and the product of inertia with respect to the plane of motion and any plane at right angles is equal to zero.

Let AD be a rigid body which has imparted to it a plane motion consisting of a rotation, relative to an axis through A, combined

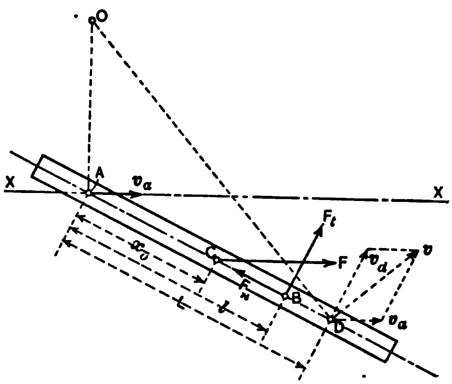

FIG. 216.

with a translation of A in a straight line XX (Fig. 216). Let C be the center of gravity of the body, W = its weight, I_a = its moment of inertia about the axis of rotation through A, $x_0 = AC$ and $l = AB$, the distance to the center of percussion of the body with respect to the axis A.

Let v_a = the velocity and a_a = the acceleration of translation of the point A, and let ω = the angular velocity and α = the angular acceleration relative to A, at the instant the body is in the position indicated.

If we let L = the distance from A to any point D, the linear velocity of the point D relative to A will be equal to

$$v_d = \omega L \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and its resultant velocity v will be the vector sum of v_d and v_a.

The instantaneous axis will be at O, the point of intersection of the perpendiculars to the vectors v_a at A and v at D, and we may write

$$\omega = \frac{v_a}{OA} = \frac{v}{OD}. \qquad (2)$$

Therefore, if any one of the velocities in equation (2) is expressed in terms of t, the others may be derived in terms of t and the distances to the instantaneous axis.

The acceleration of translation will then be equal to

$$a_a = \frac{dv_a}{dt} \qquad (3)$$

and the acceleration of rotation to

$$\alpha = \frac{d\omega}{dt}. \qquad (4)$$

The resultant of the forces required to impart the motion of rotation, relative to A, will consist of a tangential component

$$F_t = \frac{W}{g}\alpha x_0 = \frac{\alpha I_a}{lg} \text{(Art. 205)}, \qquad (5)$$

acting through B, and a normal component

$$F_n = \frac{W}{g}\omega^2 x_0, \qquad (6)$$

acting along BA.

The resultant of the forces required to impart the motion of translation will be the force

$$F = \frac{W}{g}a_a \text{ (Art. 201)}, \qquad (7)$$

acting at C, parallel to XX.

When the forces F_t, F_n and F are known, the resultant of the entire system of forces acting on the body at the given instant may be found by the method of composition of forces (Art. 51).

The connecting rod of the ordinary reciprocating engine is an illustration of a motion which may be resolved, in the above manner, into a rotation about an axis through the center of the crosshead pin and a translation with a velocity equal to that of the pin.

If the point A were to move in a curve, the force at C producing the translation would consist of a normal and tangential component which could be determined if the form of the curve and the velocity of A were known. The resultant of the entire system could then be found in the manner indicated. Where all the points in the body move in curves, however, the analysis under Case I is the more suitable one to use.

In every case of rotation about an instantaneous axis, there is at each instant a point at which the acceleration is zero. This point is called the *center of acceleration* and it does not in general coincide with the instantaneous axis. It has not the importance of the instantaneous axis and it is unnecessary to determine its position in the solution of problems.

212. Momentum Due to Combined Translation and Rotation. — When the motion of a body at any instant is resolved into a motion of rotation about the center of gravity A and a motion of translation with a velocity equal to that of the point A (Case I, Art. 211) the angular momentum due to the rotation will be equal to

$$\frac{I_0}{g}\omega, \qquad (1)$$

and the linear momentum due to the translation will be equal to

$$\frac{W}{g}v_a. \qquad (2)$$

Following the notation previously adopted, the change in the angular momentum produced by the component couple M_0 during the time dt will be represented by the expression

$$M_0\,dt = \frac{I_0}{g}d\omega \text{ (Art. 206)}, \qquad (3)$$

and the change in the linear momentum produced by the component force F_t during the time dt will be represented by the expression

$$F_t\,dt = \frac{W}{g}dv_a \text{ (Art. 202)}. \qquad (4)$$

Integrating, and letting ω_0 and ω_1 equal the angular velocities

at the beginning and at the end of the time t and v_{a_0} and v_{a_1}, the corresponding values of the linear velocity of A, we obtain

$$\int_0^t M_0\, dt = \frac{I_0}{g}(\omega_1 - \omega_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$\int_0^t F_t\, dt = \frac{W}{g}(v_{a_1} - v_{a_0}). \quad . \quad . \quad . \quad . \quad . \quad (6)$$

When M_0 and F_t are constants the above equations reduce to

$$M_0 t = \frac{I_0}{g}(\omega_1 - \omega_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and

$$F_t t = \frac{W}{g}(v_{a_1} - v_{a_0}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

When the motion at any instant is resolved into a motion of rotation about a point A which is not the center of gravity and a motion of translation with a velocity equal to that of A (Case II, Art. 211) the expression for the change in the angular momentum due to the rotation about A during the time t will be

$$\int_0^t F_t l\, dt = \frac{I_a}{g}(\omega_1 - \omega_0), \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and that for the change in the linear momentum due to the translation will be

$$\int_0^t F\, dt = \frac{W}{g}(v_{a_1} - v_{a_0}). \quad . \quad . \quad . \quad . \quad . \quad (10)$$

213. Kinetic Energy Due to Translation and Rotation Combined. — When the motion of a body at any instant is resolved into a rotation about the center of gravity A and a translation (Case I, Art. 211) the energy of rotation about the axis through A will be equal to

$$E_r = \frac{I_0}{2g}w^2 \text{ (Art. 207)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the energy due to the motion of translation will be equal to

$$E_t = \frac{W}{2g}v_a^2 \text{ (Art. 203)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The total kinetic energy of the body will therefore be equal to

$$E = E_r + E_t = \frac{I_0}{2g}\omega^2 + \frac{W}{2g}v_a^2. \quad . \quad . \quad . \quad . \quad (3)$$

Substituting $v_a = \omega x_0$, equation (3) reduces to

$$E = \frac{I_0}{2g}\omega^2 + \frac{Wx_0^2}{2g}\omega^2 = \frac{I}{2g}\omega^2, \quad . \quad . \quad . \quad . \quad (4)$$

where I = the moment of inertia of the body about the instantaneous axis.

When the motion at any instant is resolved into a motion of rotation about a point A which is not the center of gravity (Case II, Art. 211) the energy of rotation about the axis through A will be represented by the expression

$$E_{r_a} = \frac{I_a}{2g}\omega^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and the energy of translation by the expression

$$E_{t_a} = \frac{W}{2g}v_a^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence the total kinetic energy will be equal to

$$E = E_{r_a} + E_{t_a} = \frac{I_a}{2g}\omega^2 + \frac{W}{2g}v_a^2. \quad . \quad . \quad . \quad . \quad (7)$$

In either of the foregoing cases the total change in kinetic energy during any displacement of the body will evidently be equal to the total work done by the external forces acting upon it.

214. Problems — Kinetics of Rigid Bodies. — The problems involving the applications of the principles of Kinetics to rigid bodies may be divided into three general classes.

First, where all the forces acting on a freely moving body are known; and the problem is to determine the form of the path and the motion of the body in its path.

Second, where the body is constrained to move in a given manner and part of the system of forces causing its motion is

known; and the problem is to determine the remaining forces acting on the body and the effect of the system on the motion.

Third, where the body is constrained to move in a definite manner under the action of a system of forces whose points of application are known, and the problem is to determine the forces in the system.

Practically all engineering problems in Kinetics are included in the second or the third classes. In the solution of such problems the rule previously given in connection with problems in Statics (Art. 41) should be applied, that is:

In dealing with any problem involving the action of forces the first question which arises is: What are the forces to be dealt with and what are the elements to be determined?

In determining the answer to this question we shall find that in every case the lines of action of the components of the unknown forces in a system acting on a moving body at any instant can be determined from the nature of the problem, and that their magnitudes and directions will depend on the motion which is to be imparted to the body.

It is suggested that as an aid to the correct solution of problems a sketch be made in each case, showing the positions of the lines of action and directions of all the external forces, acting on the moving body, and also the resultant of the system.

In the solution of the following problems resistances due to friction will be neglected except where definite values for such resistances are given. Unless otherwise stated the effect of the force of gravity is to be allowed for in the solution of every problem.

Problem 1.

A weight of 3 tons, starting from rest, is raised 100 ft. in 4 secs. by a constant pull on the hoisting rope. Find the tension in the rope. If, at the end of 4 secs. the tension in the rope is reduced to 1 ton, how much higher will the weight rise?

Problem 2.

A weight of 2000 lbs. is being raised vertically upwards by a system of forces at a constant velocity of 5 ft. per sec. If an additional force of 200 lbs. acting vertically upwards is applied, find the distance through which the weight will be raised during the first 5 secs. after the force begins to act. If the additional force of 200 lbs. ceases to act at the end of 5 secs. find the velocity of the weight at the end of the next 10 secs. Find the total work done in raising the weight during the last 10 secs.

If a force of 300 lbs., acting vertically downward, is applied to the weight 5 seconds after the 200 lbs. force ceases to act, find the time which will elapse before the weight comes to rest.

Problem 3.

The piston, rod and head of a steam hammer weigh together 4000 lbs. If the hammer falls 3 ft. under a constant steam pressure on the piston of 2000 lbs. before striking a blow, find the energy of the blow.

Problem 4.

A steam hammer with piston weighs 4 tons. If the area of the piston is 80 sq. ins. and the average steam pressure is 80 lbs. per sq. in. and the stroke is 4 ft., what is the maximum available energy of the blow?

Problem 5.

A weight of 50 lbs., starting from rest, falls through a distance of 10 ft. before striking a helical spring. If the resistance of the spring to compression is proportional to the amount the spring is compressed, and is equal to 800 lbs. per inch, find the maximum amount the spring will compress, and the maximum pressure between the weight and the spring.

Solution. — Neglecting air resistance the weight will have a motion of translation only; and we may also assume that the entire kinetic energy acquired by the weight in falling is used up in compressing the spring; the amount of work lost in producing a permanent distortion of the weight and spring and that due to the "internal friction" of the particles in the spring being so small as to be negligible. Following these assumptions the force of gravity is the only force, other than the reaction of the spring, acting on the falling weight and the work done by gravity on the weight will be equal to the work done by the reaction of the spring in bringing the weight to rest.

Let x = the maximum amount, in inches, which the spring is compressed. Then $800\,x$ = the maximum pressure on the spring in pounds, and the work done by the spring on the weight will be equal to the product of the average pressure and the distance the spring is compressed,

$$\int F\,ds = \frac{800\,x}{2}\,x = 400\,x^2. \qquad (1)$$

The work done by gravity on the falling weight will be equal to

$$50\,(120 + x) = 6000 + 50\,x. \qquad (2)$$

Equating (1) and (2) and solving for x we have

$$x = 3.94 \text{ ins.},$$

and hence the maximum pressure between the weight and the spring will be equal to

$$800 \times 3.94 = 3152 \text{ lbs.}$$

The potential energy due to strain (Art. 175) acquired by the spring at the instant of greatest compression, would in turn be expended in forcing the weight upward, causing it to rebound from the spring. If there were no loss

due to internal friction and other causes, the weight would rise again to the height from which it originally fell.

Problem 6.

A weight of 10 lbs. falls from a height of 2 ft. and strikes a spring, deflecting it 3 ins. Find the maximum pressure between the weight and the spring, assuming that the deflection of the spring varies directly as the force exerted upon it.

Problem 7.

A weight of 40 lbs., moving horizontally with a velocity of 10 ft. per sec., strikes a helical spring whose resistance to compression is proportional to the force applied, and is equal to 200 lbs. per inch. Find the maximum distance the spring is compressed.

Problem 8.

A train is hauled up a grade of 30 ft. per mile along a straight track by an engine which is assumed to exert a constant tractive effort of 16,000 lbs. If the weight of the engine and train is 600 tons and the grade is one mile long and the velocity of the train at the bottom of the grade is 10 miles per hour, determine the following, assuming that the frictional resistance is constant and equal to 10 lbs. per ton:

(a) The velocity of the train at the top of the grade.

(b) The time taken in going up the grade.

(c) The total work done by the engine in going up the grade.

(d) The horse-power the engine exerts at the bottom of the grade; also at the top.

(e) The horse-power necessary to keep the train moving up grade at a uniform speed of 20 miles per hour.

Solution. — In this case the engine and train may be treated as a rigid body having a motion of translation in a straight line along an inclined plane, the resultant of all the forces acting being a force whose line of action coincides with the direction of the motion of the body and passes through its center of gravity.

Resolve the weight of the train into components, normal and parallel to the incline. Since the angle of slope of the incline is small the latter component will be equal to

$$F = \frac{30}{5280} \times 1{,}200{,}000 = 6820 \text{ lbs. (very nearly).}$$

The former component will be balanced by the reaction of the track on the wheels.

The "train resistance" will be assumed to be the same as on a level track, that is,

$$F_1 = 10 \times 600 = 6000 \text{ lbs.}$$

Hence the resultant of all the forces acting on the train will be equal to

$$R = 16{,}000 - F - F_1 = 3180 \text{ lbs.}$$

(a) Since R = a constant,

$$Rs = \frac{M}{2}(v_1^2 - v_0^2) \text{ (Art. 203),}$$

where $$v_0 = \frac{10 \times 5280}{3600} = \frac{44}{3} = 14.7 \text{ ft. per sec.}$$

Hence $$3180 \times 5280 = \frac{1,200,000}{2 \times 32.2}\left[v_1^2 - \left(\frac{44}{3}\right)^2\right]$$

and $$v_1 = 33.4 \text{ ft. per sec.} = 22.8 \text{ miles per hr.}$$

(b) Since R = a constant, the motion of the train will be uniformly accelerated and the distance will be equal to the product of the average velocity and the time, that is,

$$s = \frac{1}{2}(v_1 + v_0)\,t.$$

Therefore

$$t = \frac{2 \times 5280}{33.4 + 14.7} = 220 \text{ secs.} = 3 \text{ min. } 40 \text{ secs.}$$

(c) Since the tractive force is constant, the work done by the engine will be equal to

$$Fs = 16,000 \times 5280 = 84,480,000 \text{ ft.-lbs.}$$
$$= 42,240 \text{ ft.-tons.}$$

(d) The horse-power exerted by the engine may be found by the formula

$$\text{h.p.} = \frac{Fv}{550} \text{ (Art. 171).}$$

Hence at the bottom of the grade

$$\text{h. p.} = \frac{16,000 \times 14.7}{550} = 428,$$

and at the top of the grade

$$\text{h.p.} = \frac{16,000 \times 33.4}{550} = 972.$$

(e) If the train is to move at a uniform speed of 20 miles per hour, the forces acting upon it must be in equilibrium and hence the tractive force of the engine must be equal to $6000 + 6820 = 12,820$ lbs., and the power required will be

$$\text{h.p.} = \frac{12,820 \times 20 \times 5280}{60 \times 33,000} = 684.$$

Problem 9.

(a) Find the work done in starting a train weighing 400 tons from rest and increasing the speed to 40 miles per hour, in 2 minutes along a straight and level track, assuming that the tractive force of the engine is constant. Assume a constant "train resistance" of 12 lbs. per ton.

(b) Find the horse-power that the engine exerts when the speed is 20 miles per hour.

(c) Find the horse-power necessary to keep the train moving at a uniform speed of 40 miles per hour.

(d) If the power is shut off at this speed, how far will the train run before coming to rest?

Problem 10.

Solve Problem 8, assuming that the tractive effort exerted by the engine decreases uniformly as the distance increases fr.m 25,000 lbs. at the bottom of the grade to 15,000 lbs. at the top.

Problem 11.

Solve Problem 8, assuming that the train starts from rest.

Problem 12.

Assuming that, when the speed of the train in Problem 9 reaches 15 miles per hour, the horse-power exerted by the engine falls to 800 and remains constant during the next 2 minutes, find the speed of the train at the end of that time; also find the distance traveled during the two minutes. Solve by use of diagrams. (See Art. 172.)

Problem 13.

A weight of 3000 lbs. is raised 500 feet at a uniform speed by winding the hoisting rope on a drum. If weight of the rope is 2 lbs. per foot, find the work done.

Problem 14.

A weight of 2000 lbs., starting from rest, is raised 100 ft. by a force whose magnitude decreases uniformly in proportion to the distance the weight is raised. If the initial magnitude of the force is 3000 lbs. and its final magnitude 2000 lbs., find the final velocity of the weight. Plot the power diagram and find the maximum power exerted on the hoisting rope.

Problem 15.

If the initial magnitude of the force applied to the weight in Problem 14 is 3000 lbs. and the force decreases uniformly as the distance the weight is raised, find the final magnitude of the force in order that the weight shall come to rest at the height of 100 ft. Plot the power diagram and find the maximum power exerted on the hoisting rope.

Problem 16.

If a weight of 2000 lbs., starting from rest, is raised by a constant force of 3000 lbs. applied to the hoisting rope, find the point at which the power must be shut off in order that the weight shall come to rest at the height of 100 ft. Find the maximum power exerted.

Problem 17.

A perfectly flexible cord, 20 ft. long, hangs over a smooth peg. If the cord is held so that the length on one side of the peg is 11 ft. and on the other 9 ft. and is suddenly released, find the velocity of the cord at the instant the end slides off the peg, neglecting the effect of the rigidity of the cord and the friction between the cord and the peg and taking no account of the size of the peg.

Prove that the tension at the middle of the cord is always equal to the weight of the shorter of the lengths on the two sides of the peg.

Problem 18.

The weight $W = 1000$ lbs. (Fig. 217) is drawn from B to C by winding a rope around a drum at A. If the weight starts from rest at B and the pull on the rope is constant and equal to 100 lbs., find the velocity at C, neglecting friction.

Find the power exerted at the drum when the weight reaches C.

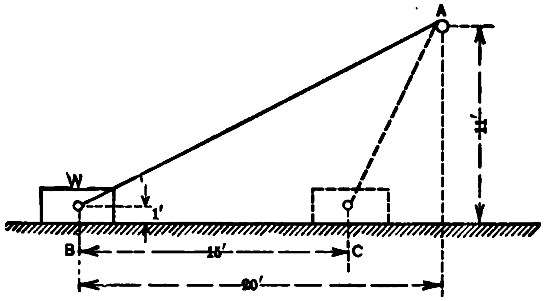

FIG. 217.

Problem 19.

Solve Problem 18, assuming a constant pull of 600 lbs. on the rope and a constant coefficient of friction between the weight and the plane which is equal to 0.25.

Find the power exerted at the drum when the weight reaches C.

Problem 20.

Plot a diagram of the work required to move the weight in Problem 18 from B to C at a uniform speed and find the work done, assuming a constant coefficient of friction equal to 0.3, between the weight and the plane.

Problem 21.

Find the work necessary to produce a speed of 300 revolutions per minute, starting from rest, in the parallel rod AB (Fig. 218) weighing 300 lbs., neglecting friction on the bearings. If the motion is uniformly accelerated and the above speed is produced in 2 minutes, find the components of the tangential accelerating force exerted at the pins A and B.

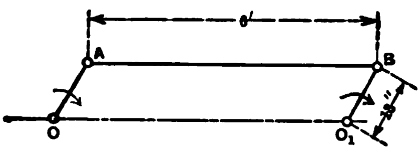

FIG. 218.

Find the pull on the centers O and O_1 due to centrifugal force when the speed is 300 revolutions per minute.

Problem 22.

Deduce the formula $e = \dfrac{Gv^2}{gR}$ for the difference in level of the rails, called the super-elevation of the outer rail, which is necessary to maintain a normal

pressure between the wheels and the track as a train goes around a curve; where G = the gage of the track, R = the radius of the curve, g = the acceleration due to gravity and v = the speed of the train.

Solution. — The motion of the train may be considered, with a slight approximation, to be one of translation only and, if A (Fig. 219) represents the center of gravity of a car, the resultant deviating force acting on the car will be a force F_n, acting through A perpendicular to a vertical axis through the center of curvature of the track.

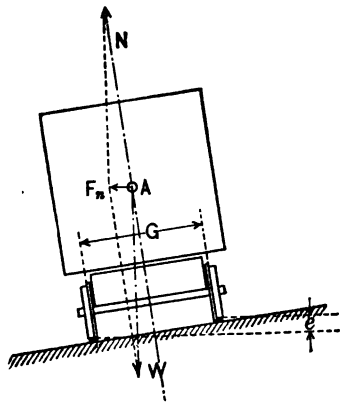

FIG. 219.

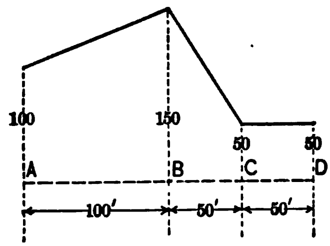

FIG. 220.

The forces acting on the car will be W, the force of gravity, N, the normal reaction of the track, and the accelerating and retarding forces acting in the direction of the motion.

The deviating force F_n will be the resultant of W and N and, since its magnitude must be equal to

$$F_n = M\frac{v^2}{r} \text{ (Art. 201),}$$

we shall have

$$W \tan\theta = \frac{Wv^2}{gR},$$

where θ = the angle of slope of a plane across the track. For a moderate angle of slope

$$\tan\theta = \sin\theta = \frac{e}{G} \text{ (very nearly).}$$

Hence
$$W\frac{e}{G} = \frac{Wv^2}{gR},$$

and
$$e = \frac{Gv^2}{gR}.$$

Problem 23.

A car weighing 2000 lbs. is hauled up an incline of 2 ft. in 100 ft. by a force whose magnitude at any point is represented by the ordinate of the force space diagram (Fig. 220). If the resistance due to friction is constant and equal to

20 lbs., find the velocity at the point D, assuming the velocity at A equal to zero. Find the horse-power exerted when the weight passes B.

Problem 24.

Plot a diagram showing the power exerted in moving the weight in Problem 23 at each point in the path.

Problem 25.

Solve Problem 23, assuming that the weight moves down the incline instead of up.

Problem 26.

A sphere, 10 in. diameter and weighing 100 lbs., is attached by a cord to an axis about which it rotates at a speed of 300 revolutions per minute. The distance from the center of the sphere to the axis is 20 ins. Neglecting the effect of the force of gravity and the weight of the cord; (a) find the tension in the cord; (b) find the kinetic energy of the weight.

Problem 27.

Make an approximate solution of Problem 26, assuming the motion of the sphere to be one of translation instead of rotation.

Problem 28.

A straight rod, 6 ft. long, of uniform section and weighing 60 lbs., revolves in a horizontal plane about a vertical axis through its middle point O, at a speed of 500 revolutions per minute. (a) Find the tension due to centrifugal force at the middle section of the rod. (b) Find the tension at a section 2 ft. from O. (c) Find the kinetic energy of the rod. (d) Find the resultant of the system of forces required to impart the speed of 500 revolutions per minute in 20 secs.

Solution. — Since the axis passes through the center of gravity, the force of gravity will tend to produce bending in the rod only and will have no effect on its motion.

(a) The tension at the middle section must be equal to the resultant deviating force acting on the part of the rod on one side of the axis and its magnitude will be equal to

$$F_n = \frac{\omega^2}{g} W x_0 \text{ (Art. 205).}$$

Since
$$\omega = \frac{2\pi\, 500}{60} = \frac{50\pi}{3} = 52.4 \text{ rad. per sec.,}$$

and the moment of the half of the rod about the axis of rotation is equal to

$$W x_0 = 30 \times \frac{3}{2} = 45 \text{ ft.-lbs.,}$$

the tension will be equal to

$$T_1 = \frac{(52.4)^2}{32.2} \times 45 = 3840 \text{ lbs.}$$

(b) The tension at a section 2 ft. from O must be equal to the resultant deviating force acting on the part of the rod between the section and the end; and, since the moment of that portion about the axis of rotation is equal to

$$W x_0 = 10 \times \frac{5}{2} = 25 \text{ ft.-lbs.,}$$

the tension will be equal to

$$T_2 = \frac{(52.4)^2}{32.2} \times 25 = 2130 \text{ lbs.}$$

(c) The kinetic energy of the rod will be equal to

$$E_r = \frac{\omega^2 I}{2g} \text{ (Art. 207).}$$

Assuming that I is the same as if the mass of the rod were concentrated along its center line we obtain by use of the formula

$$I = \frac{WL^2}{12} \text{ (Art. 135),}$$

$$I = \frac{60 \times 36}{12} = 180 \text{ lbs. (ft.)}^2.$$

Hence

$$E_r = \frac{(52.4)^2 \times 180}{64.4} = 7670 \text{ ft.-lbs.}$$

(d) Since the axis of rotation passes through the center of gravity and the rod is considered to be symmetrical with respect to the plane of rotation, the resultant of the system of external forces required to impart an acceleration α will be a couple whose moment is equal to

$$M = \frac{\alpha I}{g} \text{ (Art. 205, } \textit{Case Ia}\text{).}$$

Assuming that the motion is uniformly accelerated,

$$\alpha = \frac{\omega}{20} = \frac{5\pi}{6}$$

and

$$M = \frac{5\pi}{6} \times \frac{180}{32.2} = 14.6 \text{ ft.-lbs.}$$

Problem 29.

Deduce the expression for the tension at any cross section of the rod in Problem 28 in terms of x, the distance of the section from the axis of rotation.

Problem 30.

Solve Problem 28, assuming that the rod is constrained so as to rotate in a horizontal plane about vertical axis through one end.

Problem 31.

Assuming that the rod in Problem 28 is made to rotate at a uniform speed of 500 revolutions per minute about an axis through its middle point O, making an angle of 60° with the center line of the rod, find its kinetic energy. Compute the magnitude of the constraining couple necessary to hold the rod at an angle of 60° with the axis.

Solution. — Assuming, as in Problem 28, that the mass of the rod is concentrated along its center line we obtain from the formula

$$I = \frac{WL^2}{3} \sin^2\theta \text{ (Art. 135),}$$

$$I = 2 \times \frac{30 \times 9}{3} \times \frac{3}{4} = 135 \text{ lbs. (ft)}^2.$$

Hence

$$E_r = \frac{(52.4)^2 \times 135}{64.4} = 5760 \text{ ft.-lbs.}$$

The moment of the couple which holds the rod at an angle of 60° with its axis of rotation will be equal to

$$M_y = \Sigma M_{y_n} = \omega^2 K_{xz_m} \text{ (Art. 205, } Case\ IIa).$$

Since the weight of the rod is assumed to be concentrated along its center line, the product of inertia

$$K_{xz_m} = \frac{K_{xz}}{g} = \frac{\int xz\,dw}{g} \text{(Art. 138)}$$

may be expressed in terms of the length of the rod, as follows:

Let dw = the weight of an elementary length ds at a distance from O equal to s. Then $dw = 10\,ds$, $x = s \sin 60°$ and $z = s \cos 60°$.

Hence

$$K_{xz} = 10 \sin 60° \cos 60° \int_{-3}^{3} s^2\,ds = 45\sqrt{3} \text{ lbs. (ft.)}^2,$$

and
$$M_y = \frac{45\sqrt{3}}{32.2}(52.4)^2 = 6650 \text{ ft.-lbs.}$$

It should be noted that the preceding solution is equivalent to finding the sum of the moments about O of the deviating forces acting on the particles of the rod.

Problem 32.

Find the resultant of the system of forces required to impart to the rod in Problem 31 the speed of 500 revolutions per minute in 20 secs.

Problem 33.

A straight rod OA, of uniform section and material, rotates about a vertical axis OB through one end (Fig. 221) at a speed of 60 revolutions per minute. The diameter of the rod = 1 in., its length = 16 in. and its weight = 20 lbs. Assuming that the weight of the rod is concentrated along its center line, find the angle θ which the rod, if free to turn in the vertical plane will finally make with the axis OB. Find the H and V components of the pull exerted on the axis of rotation at O. Find the kinetic energy of the rod.

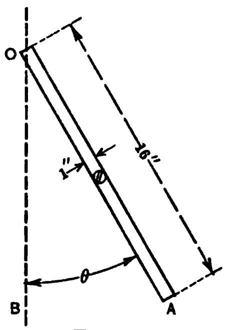

FIG. 221.

Problem 34.

Find the tension at the middle section of the rod in Problem 33.

Problem 35.

If the rod in Problem 33 were constrained to rotate at a fixed angle of 30° with OB, find the resultant of the system of forces required to impart a speed of 100 revolutions per minute in 10 secs. Find the kinetic energy of the rod and the magnitude of the couple exerted in the plane of AOB when the speed is 100 revolutions per minute.

Problem 36.

A straight rod of uniform section and material, 6 ft. long and weighing 40 lbs. (Fig. 222) is suspended from a horizontal axis through O, perpendicular to its length. If a force $F = 10$ lbs. is applied in a horizontal direction at the lower end of the rod for $\frac{1}{5}$ sec. in the plane of rotation:

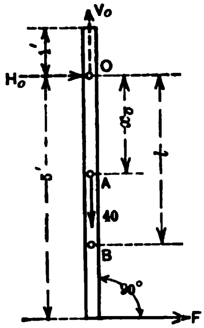

FIG. 222.

(a) Find the pressure on the axis at the instant the force is applied.

(b) Find the angle θ_1 through which the rod will swing from the vertical position before stopping.

Solution. — (a) Let H_0 = the horizontal component and V_0 = the vertical component of the pressure exerted on the rod at the axis. We will assume that the mass of the rod is concentrated along its center line, in which case the distance of the center of percussion B from O will be equal to

$$l = \frac{I}{Wx_0} = \frac{I_0}{Wx_0} + x_0 \text{ (Art. 205).}$$

$$I_0 = \frac{WL^2}{12} = \frac{40 \times 36}{12} = 120 \text{ lbs. (ft.)}^2 \text{ (Art. 135),}$$

and $\qquad x_0 = 2$ ft.

Hence $\qquad l = \frac{120}{2 \times 40} + 2 = 3.5$ ft.

Since the resultant of all the forces acting on the rod will pass through its center of percussion (Art. 205), the force exerted by the axis O must be of such magnitude and direction that the algebraic sum of the moments of the forces about B will be equal to zero, and hence

$$3.5\,H_0 = 10 \times 1.5,$$

and $\qquad H_0 = 4\frac{2}{7}$ lbs.,

acting in the direction indicated (Fig. 222). At the instant the force F is applied the algebraic sum of the vertical components will be equal to zero and hence

$$V_0 = 40 \text{ lbs., acting upward.}$$

The resultant pressure of the axis on the rod will be equal to

$$R_0 = \sqrt{(H_0)^2 + (V_0)^2} = 40.2 \text{ lbs.,}$$

whose line of action can readily be determined.

(b) Since the force F acts for $\frac{1}{5}$ sec. only, we will assume that the motion of the rod during that time is so small that the line of action of F remains unchanged and the retarding effect of the force of gravity can be neglected.

Then $\qquad M = 5\,F = \dfrac{\alpha I}{g}$ (Art. 205),

where $\qquad I = I_0 + Wx_0^2 = 120 + 40 \times 4 = 280$ lbs. (ft.)^2

and $\qquad \alpha = \dfrac{5 \times 10 \times 32.2}{280} = 5.75$ rad. per sec.^{2}

Neglecting the effect of gravity, the angular velocity at the end of $\frac{1}{5}$ sec. will be equal to

$$\omega = \alpha t = \frac{5.75}{5} = 1.15 \text{ rad. per sec.}$$

and the kinetic energy will be equal to

$$E = \frac{\omega^2 I}{2g} = \frac{(1.15)^2 \times 280}{64.4} = 5.75 \text{ ft.-lbs.}$$

Since the force of gravity is the only force retarding the motion of the rod, the work done by gravity as the rod swings through the angle θ_1 will be equal to E. When the rod is displaced at any angle θ from its mid-position the moment of its weight about O will be equal to

$$M = 40 \times 2 \sin\theta = 80 \sin\theta.$$

Hence
$$E = 5.75 = \int_0^{\theta_1} M\,d\theta = 80 \int_0^{\theta_1} \sin\theta\,d\theta$$
$$= 80\,(1 - \cos\theta_1).$$

Therefore
$$\cos\theta_1 = 0.928$$

and
$$\theta_1 = 21^\circ\,52'.$$

Problem 37.

Find the angle through which the rod in Problem 36 will swing from the vertical position if the force F acts through an angle of 4°.

Problem 38.

If the rod in Problem 36 is held at an angle of 60° with the vertical and is suddenly released and allowed to swing down under the action of gravity only, find its angular velocity when it reaches the vertical position. Find the force exerted on the rod at the axis, when it reaches the vertical position.

Problem 39.

Solve Problem 38 assuming a constant resistance due to friction at the axis, the moment of which is equal to 20 in.-lbs.

Problem 40.

The cylinder A, weighing 10 lbs., is connected to the sphere C, weighing 20 lbs., by a slender rod, weighing 5 lbs., which is suspended from a horizontal axis through O (Fig. 223). Find the time of a single oscillation, assuming no friction on the axis. Find the angle through which the rod may be allowed to swing without causing an error of more than 1 part in 1000 in the value of the time as calculated by the formula $t = \pi \sqrt{\dfrac{L}{g}}$ (Art. 156).

Problem 41.

Indicate an experimental method for determining the moment of inertia of the weight (Fig. 223): (a) about the axis of the cylinder A; (b) about a parallel axis through the center of gravity.

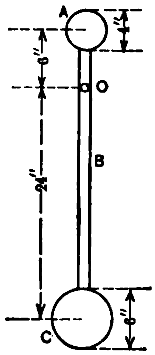

FIG. 223.

Problem 42.

A wheel, 4 ft. diameter and weighing 150 lbs., is mounted on a shaft and has imparted to it a speed of 300 revolutions per minute in 1 minute by a constant horizontal force F, applied tangent to the rim. The axis of the wheel is horizontal and the moment of inertia about the axis is 500 lbs. (ft.)^2. Assuming no friction at the shaft:

(a) Find the magnitude of the force F.

(b) Find the pressure of the bearings on the shaft.

(c) Find the kinetic energy at the end of 1 minute, assuming that the wheel starts from rest.

(d) Assuming that, when the speed is 300 revolutions per minute, the force F ceases to act and a brake is applied to the shaft which exerts a force of friction whose moment is equal to 200 in.-lbs., find the number of revolutions the wheel will make before stopping.

Solution. — (a) The sum of the moments of the external forces acting on the wheel, about the axis of rotation, will be equal to

$$M = \frac{\alpha I}{g} \text{ (Art. 205, } \textit{Case Ia}\text{)},$$

where
$$\alpha = \frac{\omega}{t} = \frac{2\pi\,300}{60 \times 60} = \frac{\pi}{6} \text{ rad. per sec.}^2$$

and
$$M = 2F.$$

Hence
$$F = \frac{\pi\,500}{2 \times 6 \times 32.2} = 4.07 \text{ lbs.}$$

(b) Since the axis of rotation passes through the center of gravity and the wheel is symmetrical with respect to the plane of rotation, the resultant of all the external forces acting on the wheel will be the couple $M = 2F$.

Hence the horizontal component of the pressure exerted on the shaft will be equal to

$$H_0 = 4.07 \text{ lbs.},$$

acting in a direction opposite to F, and the vertical component will be an upward force

$$V_0 = 150 \text{ lbs.},$$

equal to the weight.

(c) The angular velocity of the wheel at the end of 60 secs. will be equal to

$$\omega = \frac{\pi}{6} \times 60 = 10\pi \text{ rad. per sec.}$$

and the kinetic energy (Art. 207) will be equal to

$$E = \frac{\omega^2 I}{2g} = \frac{100\pi^2\,500}{64.4} = 7660 \text{ ft.-lbs.}$$

(*d*) The work done by the retarding couple must be equal to the kinetic energy.

Hence $$E = M\theta = M\,2\pi N,$$

where N = the number of revolutions the wheel makes before stopping. Solving for N we obtain

$$N = \frac{7660 \times 12}{2\pi\,200} = 73.$$

Problem 43.

Find the kinetic energy of the flywheel (Problem 3, Art. 137) when rotating at a speed of 120 revolutions per minute. Find the change in kinetic energy due to a loss in speed equal to 2 per cent.

Problem 44.

A solid cylinder, 4 ft. diameter, weighing 600 lbs., rotates on its axis at a speed of 100 revolutions per minute. Assuming that the moment of the friction on the bearings is 20 in.-lbs., find the force which, if applied tangent to the rim, will increase the speed to 300 revolutions per minute in 30 secs. Find the increase in kinetic energy. If the force tangent to the rim ceases to act when the speed is 300 revolutions per minute, find the time which will elapse before the wheel comes to rest.

Problem 45.

A wheel, weighing 4000 lbs., rotates in a horizontal plane about a vertical axis through its center at a speed of 500 revolutions per minute. If its center of gravity is $\frac{1}{4}$ in. from the axis, find the pressure on the axis due to centrifugal force.

Problem 46.

Two pulleys mounted on a length of shaft are subjected to the constant belt pulls shown (Fig. 224). If the weight of the pulleys and shaft combined

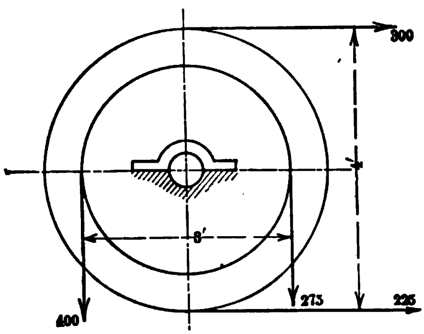

FIG. 224.

is equal to 600 lbs. and the moment of inertia = 2000 lbs. (ft.)^2, find the speed which, if the shaft starts from rest, will be imparted in 1 minute, assuming no friction on the bearings. Find the resultant pressure on the bearings.

Problem 47.

If the shaft shown (Fig. 224) rotates at a uniform speed of 400 revolutions per minute under the action of the forces shown, find the friction loss in horsepower.

Problem 48.

Two spheres A and B, weighing 10 lbs. and 50 lbs., respectively, connected by a rod weighing 10 lbs., rotate in a horizontal plane about a vertical axis through O at a uniform speed of 400 revolutions per minute (Fig. 225). Neglecting the effect of gravity, find the magnitude and direction of the force exerted on the axis.

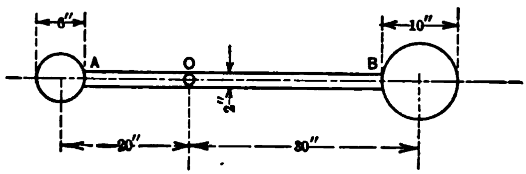

FIG. 225.

Problem 49.

Find the resultant of the system of forces required to reduce the speed of the weight in Problem 48 to 200 revolutions per minute in 20 seconds, neglecting the effect of gravity.

Problem 50.

Assuming that the weight in Problem 48 is free to turn about O in the vertical plane as well as the horizontal, find the angle which its center line will finally assume with the axis of rotation when the speed is 100 revolutions per minute, if the effect of gravity is taken into account.

Problem 51.

A thin ring 6 ft. diameter rotates about an axis through its center perpendicular to its plane at a speed of 500 revolutions per minute. If the weight of the ring is equal to 100 lbs. find the tension due to centrifugal force. (Refer to Art. 191, dealing with the effect of centrifugal force on the tension in a belt.)

Problem 52.

Find the kinetic energy of the ring in Problem 51 and determine the resultant of the system of forces necessary to impart the speed in 50 seconds.

Problem 53.

Prove that the intensity of the tensile stress on the cross section of a thin ring, or cylinder, due to centrifugal force is independent of the area of the section.

Problem 54.

Three cylinders, A, B and C, each 8 in. diameter and weighing 100 lbs., are connected by slender rods along the lines joining their centers of gravity, which are located in the same plane, perpendicular to the axes of the cylinders (Fig. 226). If the system rotates about an axis through its center of gravity O, perpendicular to the above-mentioned plane, at a speed of 300 revolutions per minute, find the tension in the rods, neglecting the effect of gravity and the weight of the rods. Find the kinetic energy of the system of weights.

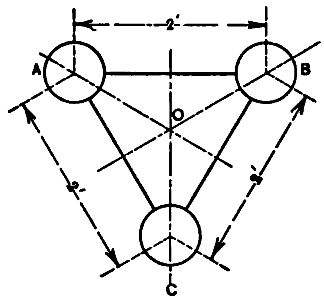

FIG. 226.

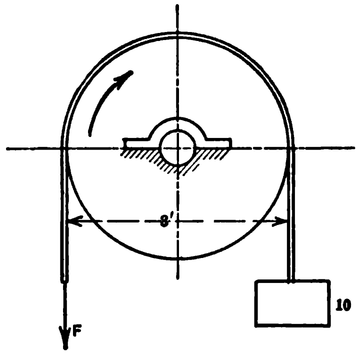

FIG. 227.

Problem 55.

Find the kinetic energy of the weights in Problem 54, assuming that the motion of each one is translation only.

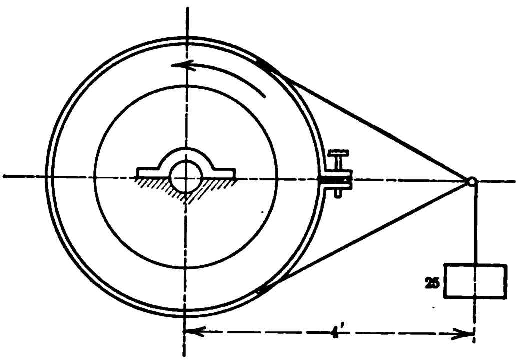

FIG. 228.

Problem 56.

A "band brake," consisting of a pulley 3 ft. diameter, rotating under a stationary band (Fig. 227) is used for absorbing and measuring power. If the pulley turns at a uniform speed of 600 revolutions per minute and the force $F = 20$ lbs., find the horse-power measured by the brake, neglecting the friction at the bearings of the shaft.

Problem 57.

Find the coefficient of friction between the belt and the pulley in Problem 56 according to the theory of belting (Art. 191).

Problem 58.

A Prony brake for absorbing and measuring power (Fig. 228) is constructed in such a manner that the tension in the band and hence the moment of the friction between the band and the pulley can be regulated by means of an adjusting screw, so that it will always balance the moment of a weight about the axis of the pulley.

If the pulley rotates at a uniform speed of 400 revolutions per minute and the weight on the brake is 25 lbs., find the horse-power measured.

Problem 59.

A wheel, 8 ft. diameter, starting from rest, is rolled up the inclined plane (Fig. 229) by a constant force F applied to the rim in a direction parallel to

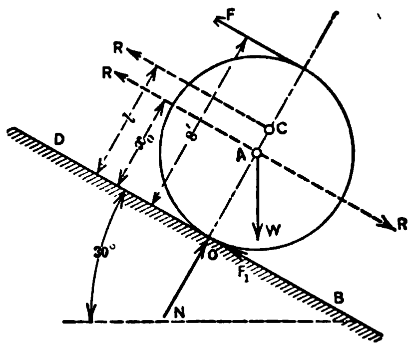

FIG. 229.

the plane. The weight of the wheel is $W = 200$ lbs.; the moment of inertia about its axis A is $I_0 = 800$ lbs. (ft.)^2. If $F = 80$ lbs. and the sliding friction at O, the element of contact with the plane, is sufficient to prevent slipping, determine:

(a) The force exerted by friction at O; (b) the distance the wheel will move in 5 secs.; (c) the linear velocity of its center of gravity at the end of 5 secs.; (d) the kinetic energy of the wheel at the end of 5 secs.; (e) the work done by the force F during 5 secs. Assume that the effect of rolling friction is so small that it may be neglected.

Solution. — In this case the body has a plane motion of rotation about an instantaneous axis which is always the element of contact between the wheel and the plane. The problem may be solved by resolving the motion of the wheel into a motion of rotation about an axis through its center of gravity A, parallel to the instantaneous axis O, and a motion of translation.

Since A moves in a straight line, the resultant of all the external forces acting on the wheel may be resolved into a single force, acting through A,

parallel to BD, producing the motion of translation, and a couple producing the rotation. (Art. 211, *Case Ia.*)

The forces acting on the wheel will be the force $F = 80$ lbs., the force $W = 200$ lbs., the normal component N, of the reaction of the plane at O, and the force F_1, due to the friction between the wheel and the plane at O.

If we resolve these forces into components parallel and perpendicular to BD, the components perpendicular to BD will balance, since the motion of the center of gravity is rectilinear.

Hence the resultant of all the external forces will be a force parallel to BD and equal to

$$R = F + F_1 - W \sin 30°,$$

and its line of action will pass through C, the center of percussion with respect to the axis O.

Since the distance x_0 is constant, the distance OC will be equal to

$$l = \frac{I_0}{Wx_0} + x_0 = \frac{800}{800} + 4 = 5 \text{ ft.}$$

Since the sum of the moments of all the forces acting on the wheel will be equal to the moment of their resultant about the axis at O,

$$R \times 5 = 8 \times 80 - 4 \times 200 \sin 30°$$

and

$$R = 48 \text{ lbs.}$$

(a) Substituting in the first equation the values of R, F and W we have

$$48 = 80 + F_1 - 200 \sin 30°$$

and

$$F_1 = 68 \text{ lbs.}$$

(b) Resolving R into a couple M and an equal and parallel force at the center of gravity, we have

$$R = 48 = \frac{W}{g} a_a = \frac{200}{32.2} a_a$$

and

$$a_a = 7.73 \text{ ft. per sec.}^2$$

Hence the distance traveled in 5 secs., will be equal to

$$s = \frac{1}{2} a_a t^2 = \frac{1}{2} \times 7.73 \times 25 = 96.6 \text{ ft.}$$

(c) The linear velocity of the center of the wheel at the end of 5 secs. will be equal to

$$v_a = a_a t = 7.73 \times 5 = 38.6 \text{ ft. per sec.}$$

(d) The kinetic energy of the wheel will be equal to

$$E = \frac{\omega^2 I_0}{2g} + \frac{v_a^2 W}{2g} \text{ (Art. 213).}$$

The angular velocity about the center at any instant will be the same as the angular velocity about the instantaneous axis and, at the end of 5 secs.,

$$\omega = \frac{v_a}{r} = \frac{38.6}{4} = 9.66 \text{ rad. per sec.}$$

Hence,

$$E = \frac{(9.66)^2 \times 800}{64.4} + \frac{(38.6)^2 \times 200}{64.4}$$

$$= 1160 + 4640 = 5800 \text{ ft.-lbs.}$$

Or, the kinetic energy may be found from the equation

$$E = M\theta + Rs,$$

where the total angle through which the wheel turns in traveling the distance s will be equal to

$$\theta = \frac{s}{r} = \frac{96.6}{4} = 24.15 \text{ rad.}$$

and

$$M = R \times 1 = 48 \text{ ft.-lbs.}$$

Hence,

$$E = 48 \times 24.15 + 48 \times 96.6 = 5800 \text{ ft.-lbs.}$$

(e) Since the velocity of the point of application of the force F is always equal to 2 times the velocity of the center of the wheel, the distance through which the force F moves, while the center of the wheel travels a distance s, will be equal to $2s$. Hence the work done by the force F will be equal to

$$F \times 2s = 80 \times 2 \times 96.6 = 15{,}460 \text{ ft.-lbs.}$$

Or, the work may be determined by adding to the kinetic energy the work done in raising the wheel against the force of gravity through a distance equal to 96.6 sin 30° = 48.3 ft., that is, the total work done by the force F will be equal to

$$5800 + 200 \times 48.3 = 15{,}460 \text{ ft.-lbs.}$$

Problem 60.

A solid cylinder, 5 ft. diameter, weighing 2000 lbs., is rolled along a horizontal plane by a constant horizontal force, F = 200 lbs. applied at its axis. Assuming no slipping and neglecting rolling friction, if the cylinder starts from rest, how far will it roll in 10 secs.? Find its kinetic energy at the end of 10 secs.

Problem 61.

Solve Problem 60 assuming the weight to be a solid sphere of the same diameter and total weight.

Problem 62.

Solve Problem 60 assuming the cylinder has an initial velocity of 20 ft. per sec.

Problem 63.

A wheel, 4 ft. diameter, weighing 400 lbs., is rolled along a horizontal plane by a constant horizontal force, F = 20 lbs., applied to a cord wound around the circumference of a drum on the wheel which is 2 ft. in diameter (Fig. 230). The moment of inertia of the wheel about its axis O is equal to I_0 = 1200 lbs.

(ft.)^2. Assuming no slipping and neglecting rolling friction, find the time taken in rolling the wheel 100 ft., starting from rest. Find the kinetic energy of the wheel after rolling 100 ft.

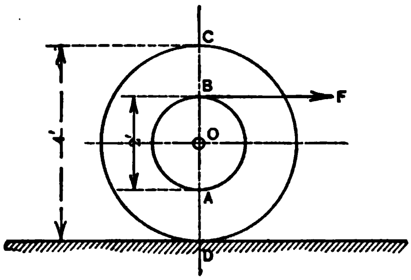

FIG. 230.

Problem 64.

Solve Problem 63: (a) Assuming that an additional force, equal and parallel to F and acting in the same direction, is applied in a similar manner to the drum at A; (b) Assuming that an additional force, equal and parallel to F and acting in the opposite direction, is applied at A.

Problem 65.

Solve Problem 63: (a) Assuming that an additional horizontal force of 10 lbs. is applied at C in the same direction as F to a cord wound around the rim of the wheel; (b) Assuming a horizontal force of 10 lbs. is applied at C in the opposite direction to F.

Problem 66.

Solve Problem 60, assuming that there is no friction between the wheel and the plane.

Problem 67.

Solve Problem 63, assuming that there is no friction between the wheel and the plane.

NOTE. In this case the resultant of the system of forces acting on the wheel will be the horizontal force F whose magnitude, direction and line of action are known. By resolving the resultant into a couple and a single force acting at O, the two components of the resultant motion, viz.: rotation about O and translation, can be determined.

Problem 68.

A flat circular disc, 2 ft. diameter, weighing 80 lbs., rests on its face on a horizontal plane. If a constant force of 5 lbs. is applied to a flexible string, wrapped around the circumference and free to unwind as the disc moves, determine the direction and acceleration of the motion of the center of the disc, assuming that it starts from rest and that there is no friction between it and the plane. Find the kinetic energy of the disc at the end of 5 secs.

Problem 69.

Solve Problem 68, assuming that the disc is replaced by a thin ring, 2 ft. diameter, weighing 20 lbs.

Problem 70.

A solid cylinder of radius r and weight W, starting from rest at the top, rolls without slipping down an inclined plane 10 ft. long, making an angle of 30° with the horizontal. When the cylinder reaches the foot of the incline determine: (a) the linear velocity of its center; (b) its kinetic energy; (c) the time taken in rolling down the plane; (d) the required coefficient of friction between the cylinder and the plane.

Problem 71.

Solve Problem 70, substituting a thin hollow cylinder, of radius r and weight W, for the solid cylinder.

Problem 72.

Solve Problem 70, substituting for the solid cylinder: (a) a solid sphere of radius r and weight W; (b) a thin hollow sphere of radius r and weight W.

Problem 73.

A wheel A (Fig. 231), 2 ft. diameter, is mounted on an arm AC and rolls without slipping around the fixed wheel C, 6 ft. diameter, under the action of a constant force $F = 20$ lbs. acting at the center A, perpendicular to AC. The weight of $A = 100$ lbs. and its moment of inertia about the axis through its center of gravity $= 50$ lbs. (ft.)^2.

If the wheel starts from rest, find the speed of its center at the end of 30 seconds. Find its kinetic energy and the pull in the arm AC at the end of that time. Find the force of friction exerted at the point of contact O. Neglect the weight of the arm.

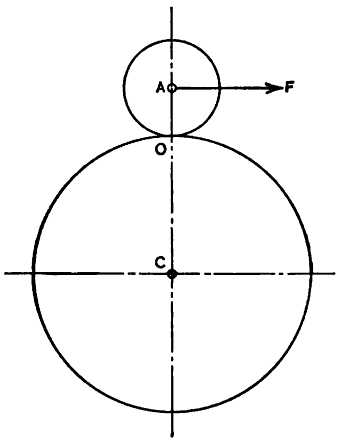

FIG. 231.

Problem 74.

A solid sphere, 1 ft. diameter, weighing 100 lbs., rolls down the inside of a cylindrical surface of 6 ft. radius. If the sphere starts from rest at a point on the arc which is 60° from the vertical through the center, find its velocity of translation when it reaches the vertical. Find its kinetic energy and the pressure on the surface at that point.

Problem 75.

If the crank DE (Fig. 232) turns at uniform speed of 300 revolutions per minute, find the magnitude of the force P, which must be exerted at the crosshead pin A, to produce the required acceleration in the connecting rod AE when the angle $\theta = 60°$. The weight of the rod is $W = 500$ lbs., its moment of inertia about the crosshead pin A is $I = 6000$ lbs. (ft.)^2; and $AE = 5$ ft., $DE = 1$ ft., and AC, the distance of the center of gravity of the rod from A, is equal to 3 ft.

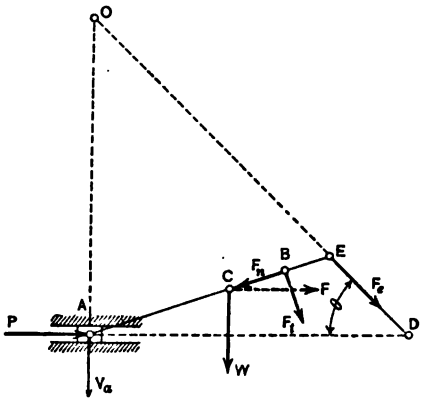

FIG. 232.

Solution. Neglecting friction, the external forces acting on the rod when in the position indicated will be the horizontal accelerating force P, exerted at the crosshead pin A, a vertical force V_a at A, the force of gravity at C, and a force F_e at E, acting along the crank DE.

In this case the resultant motion of the rod will be one of rotation about an instantaneous axis through O.

The motion may be resolved into a motion of rotation about A and translation in a horizontal direction and the resultant of the forces required to produce each component of the motion determined separately (Art. 211).

Let $AE = L$, $DE = r$, $AC = x_0$, $v_a =$ the linear velocity of A and $\omega =$ the angular velocity of AE about the instantaneous axis.

Let $\omega_1 =$ the angular velocity of DE and $t =$ the time taken in turning through the angle θ from the dead point.

Then $$\theta = \omega_1 t \quad \text{and} \quad \omega_1 = \frac{300}{60} 2\pi = 10\pi \text{ rad. per sec.}$$

We will first determine the resultant of the forces required to produce the motion of translation, which will be a horizontal force, acting through the center of gravity C, whose magnitude will be equal to

$$F = \frac{W}{g} a_a. \qquad (1)$$

The acceleration of the motion of translation may be found as follows: Let s = the distance through which the crosshead moves while the crank turns through the angle $\theta = \omega_1 t$ from the dead point: then

$$s = r - r\cos\omega_1 t + L - \sqrt{L^2 - r^2\sin^2\omega_1 t},$$

$$v_a = \frac{ds}{dt} = \omega_1 r\sin\omega_1 t + \frac{\omega_1 r^2 \sin\omega_1 t\cos\omega_1 t}{\sqrt{L^2 - r^2\sin^2\omega_1 t}},$$

and

$$a_a = \frac{dv}{dt} = \omega_1{}^2 r\cos\omega_1 t + \frac{\omega_1{}^2 r^4\sin^2\omega_1 t\cos^2\omega_1 t}{(L^2 - r^2\sin^2\omega_1 t)^{\frac{3}{2}}}$$
$$+ \frac{\omega_1{}^2 r^2(\cos^2\omega_1 t - \sin^2\omega_1 t)}{(L^2 - r^2\sin^2\omega_1 t)^{\frac{1}{2}}}.$$

Substituting $\theta = \omega_1 t$ and reducing, we have

$$a_a = \omega_1{}^2 r\left[\cos\theta + \frac{r^3\sin^2 2\theta}{4(L^2 - r^2\sin^2\theta)^{\frac{3}{2}}} + \frac{r\cos 2\theta}{(L^2 - r^2\sin^2\theta)^{\frac{1}{2}}}\right]. \quad . \quad . \quad . \quad (2)$$

When $\theta = 60°$ we obtain from equation (2)

$$a_a = \omega_1{}^2\left[0.5 + \frac{0.75}{4(25 - 0.75)^{\frac{3}{2}}} - \frac{0.5}{(25 - 0.75)^{\frac{1}{2}}}\right] = 395 \text{ ft. per sec.}^2,$$

and substituting this value in equation (1),

$$F = \frac{500}{32.2} \times 395 = 6130 \text{ lbs.}$$

The resultant of the forces required to produce the motion of rotation about A will be a force acting through B, the center of percussion of the rod with respect to A, which may be resolved into a tangential component F_t, and a deviating force F_n.

The tangential component will be equal to

$$F_t = \frac{\alpha I}{lg} \text{ (Art. 211)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where

$$l = AB = \frac{I}{Wx_0},$$

and the deviating force will be equal to

$$F_n = \frac{W}{g}\omega^2 x_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The angular velocity of AE about the instantaneous axis O will be equal to

$$\omega = \frac{\omega_1 r}{OE} = \frac{\omega_1 r\cos\theta}{\sqrt{L^2 - r^2\sin^2\theta}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Since this is also the angular velocity of AE about A, the angular acceleration α, of the motion of rotation about A, may be obtained by substituting $\omega_1 t = \theta$ and differentiating as follows:

$$\alpha = \frac{d\omega}{dt} = \frac{\omega_1{}^2 r^3\cos^2\omega_1 t\sin\omega_1 t}{(L^2 - r^2\sin^2\theta)^{\frac{3}{2}}}$$
$$- \frac{\omega_1{}^2 r\sin\omega_1 t}{(L^2 - r^2\sin^2\theta)^{\frac{1}{2}}}.$$

Substituting $\theta = \omega_1 t$ and reducing we have

$$\alpha = \omega_1^2 r \frac{\sin\theta}{\sqrt{L^2 - r^2\sin^2\theta}}\left[\frac{r^2 - L^2}{L^2 - r^2\sin^2\theta}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

When $\theta = 60°$ we obtain from equation (5)

$$\omega = \frac{\omega_1 \times 0.5}{\sqrt{25 - 0.75}} = 3.19 \text{ rad. per sec.,}$$

and from equation (6)

$$\alpha = \omega_1^2 \frac{0.866}{\sqrt{25 - 0.75}}\left[\frac{1 - 25}{25 - 0.75}\right] = -172 \text{ rad. per sec.}^2.$$

Substituting this value in equation (3) and also the value

$$l = \frac{I}{Wx_0} = \frac{6000}{500 \times 3} = 4 \text{ ft.,}$$

we obtain,

$$F_t = \frac{\alpha I}{lg} = \frac{172 \times 6000}{4 \times 32.2} = 8010 \text{ lbs.,}$$

a force acting through B, perpendicular to AE and in the downward direction, since α is a negative quantity.

Substituting the value of ω in equation (4) we obtain

$$F_n = \frac{W}{g}\omega^2 x_0 = \frac{500 \times (3.19)^2 \times 3}{32.2} = 474 \text{ lbs.,}$$

acting along BA towards A.

The resultant of all the external forces acting on the rod will be the resultant of the forces F, F_t and F_n, and moment of the resultant about A will be equal to

$$\Sigma M_a = F_t \times 4 + F \times \frac{3}{5} r \sin\theta$$
$$= 8010 \times 4 + 6130 \times \frac{3}{5} \times 0.866 = 35{,}230 \text{ ft.-lbs.}$$

Since ΣM_a will be equal to the sum of the moments about A of the external forces acting on the rod, we shall have

$$35{,}230 = F_e (AD) \sin\theta + W \frac{3}{5}(AD - r\cos\theta)$$
$$= F_e (r\cos\theta + \sqrt{L^2 - r^2\sin^2\theta})\sin\theta + W\frac{3}{5}\sqrt{L^2 - r^2\sin^2\theta}$$
$$= F_e \times 5.424 \times 0.866 + 500 \times 0.6 \times 4.924,$$

and $$F_e = \frac{35{,}230 - 1480}{4.70} = 7180 \text{ lbs.,}$$

acting along DE towards D.

Resolving the forces acting on the rod into horizontal and vertical components, the sum of the horizontal components will be equal to

$$\Sigma H = P + F_{e_h} = F + F_{t_h} - F_{n_h}.$$

Hence $$P + 7180 \times 0.5 = 6130 + 8010\,\frac{0.866}{5} - 474 \times \frac{4.92}{5} = 7050$$

and $$P = 3460 \text{ lbs.}$$

The sum of the vertical components will be equal to

$$\Sigma V = -V_a - W - F_{e_v} = -F_{t_v} - F_{n_v}.$$

Hence $\quad -V_a - 500 - 7180 \times 0.866 = -8010 \times \frac{4.924}{5} - 474 \times \frac{0.866}{5} = -7970$

and $\quad V_a = 1250$ lbs,

acting downward.

Problem 76.

Determine the magnitude of the force R which must be exerted at the crank pin E, in a direction perpendicular to DE, to produce the acceleration in the connecting rod AE (Problem 75).

Solution.— In this case the external forces acting on the rod, neglecting friction, will be the accelerating force R, a vertical force V_a at A, the force of gravity at C and a force F_e at E, acting along DE (Fig. 232).

Since the resultant of these forces will be the same as the resultant of the forces acting in Problem 75, by taking moments about E we shall obtain

$$-V_a\sqrt{L^2 - r^2\sin^2\theta} - W\frac{2}{5}\sqrt{L^2 - r^2\sin^2\theta} = -F_t \times 1 - F\frac{2}{5}r\sin\theta$$

and $-V_a \times 4.924 - 500 \times 0.4 \times 4.924 = -8010 - 6130 \times 0.4 \times 0.866 = -10{,}130$, and solving for V_a,

$$V_a = 1860 \text{ lbs. (acting downward).}$$

Resolving the forces into horizontal and vertical components, and noting that ΣH and ΣV will be the same as in Problem 75, we have

$$\Sigma H = 7050 = F_e\cos\theta + R\sin\theta,$$

and $$\Sigma V = -7970 = -F_e\sin\theta + R\cos\theta - W - V_a.$$

Substituting $\theta = 60°$ and solving these equations simultaneously, we obtain

$$R = 3300 \text{ lbs. (acting toward the right)}$$

and $$F_e = 8380 \text{ lbs. (acting toward } D).$$

It should be noted that, since the moment about O of the resultant of the forces acting on the rod must be the same in both Problems 75 and 76,

$$P \times AO = R \times EO$$

and, if the distances AO and EO are computed for any crank angle, the relative magnitudes of P and R may be obtained.

Problem 77.

Solve Problems 75 and 76 for values of θ differing by 30° for a complete revolution of the crank.

Problem 78.

The weights A and B are attached to flexible cords, wound around the two steps of the pulley shown (Fig. 233). The weight $A = 20$ lbs. and $B = 50$ lbs., and the moment of inertia of the pulley, about its axis is equal to $I = 1200$ lbs. (in.)^2.

Assuming a constant friction on the shaft, the moment of which is equal to 10 in.-lbs., and neglecting the weight of the cords, find: (a) the acceleration

of each of the weights A and B; (b) the velocities of the weights A and B at the end of 5 secs., assuming that the weights start from rest; (c) the kinetic energy of the system at the end of 5 secs., assuming that the weights start from rest.

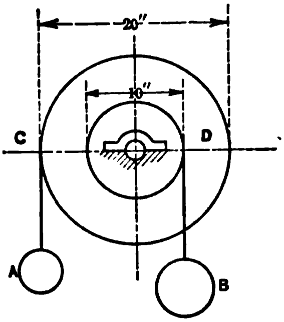

FIG. 233.

Solution. — In this case we have a system of weights, two of which, A and B, have a motion of translation, while the wheel rotates about a fixed axis through its center of gravity. Considering each weight separately, the forces acting on the weight A will be the downward force of 20 lbs., due to gravity, and the upward force T_a due to the pull exerted by the cord AC; the force acting on the weight B will be the downward force of 50 lbs., due to gravity, and the upward pull T_b exerted by the cord BD; the forces acting on the wheel, in addition to the force of gravity, will be the downward forces T_a and T_b, due to the pulls exerted by the cords at C and D, respectively, and the resultant force exerted by the bearings on the shaft, which may be resolved into a vertical component and the force of friction acting tangent to its surface (Art. 186). If we compute the moments about the center of the wheel of the forces acting on the weights, it is evident that the wheel will turn in the right-handed direction.

If we consider the velocities and accelerations of both of the weights and the wheel to be positive quantities and let v_a = the linear velocity of A, v_b = the linear velocity of B, ω = the angular velocity of the wheel, a_a = the linear acceleration of A, a_b = the linear acceleration of B and α = the angular acceleration of the wheel, at any instant, we shall have

$$\omega = \frac{12\,v_a}{10} = \frac{12\,v_b}{5} \quad \ldots\ldots\ldots\ldots \quad (1)$$

and

$$\alpha = \frac{12\,a_a}{10} = \frac{12\,a_b}{5}. \quad \ldots\ldots\ldots\ldots \quad (2)$$

The resultant of the forces acting on the weight A will be a vertical force acting through its center of gravity, which is equal to

$$T_a - 20 = \frac{20}{g}a_a \text{ (Art. 201)}, \quad \ldots\ldots\ldots \quad (3)$$

and the resultant of the forces acting on the weight B will be a vertical force acting through its center of gravity, which is equal to

$$50 - T_b = \frac{50}{g}a_b \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

The resultant of the forces acting on the wheel will be a couple, whose moment is equal to

$$T_b \times \frac{5}{12} - T_a \times \frac{10}{12} - \frac{10}{12} = \frac{1200}{144\,g}\alpha \text{ (Art. 205)}. \quad \ldots\ldots \quad (5)$$

Substituting

$$a_a = \frac{10}{12}\alpha \quad \text{and} \quad a_b = \frac{5}{12}\alpha$$

in equations 3, 4 and 5 and solving simultaneously, we obtain

$$\alpha = \frac{48\,g}{445} = 3.47 \text{ rad. per sec.}^2,$$

and $$T_a = 20 + \frac{200}{12\,g}\alpha = 21.8 \text{ lbs.},$$

and $$T_b = 50 - \frac{250}{12\,g}\alpha = 47.8 \text{ lbs.}$$

(a) From equation (2) we obtain

$$a_a = \frac{10}{12}\alpha = \frac{8\,g}{89} = 2.89 \text{ ft. per sec.}^2$$

and $$a_b = \frac{5}{12}\alpha = \frac{4\,g}{89} = 1.45 \text{ ft. per sec.}^2.$$

(*b*) Since the weights start from rest, at the end of 5 secs.

$$v_a = a_a t = \frac{40\,g}{89} = 14.47 \text{ ft. per sec.}$$

and $$v_b = \frac{1}{2}v_a = \frac{20\,g}{89} = 7.24 \text{ ft. per sec.}$$

(c) The total kinetic energy E of the system will be equal to the sum of the energies of the weights A and B and the wheel.
Hence

$$E = \frac{20}{2\,g}\left(\frac{40\,g}{89}\right)^2 + \frac{50}{2\,g}\left(\frac{20\,g}{89}\right)^2 + \frac{1200}{2 \times 144\,g}\left(\frac{48\,g}{89}\right)^2$$
$$= \frac{400\,g}{89} = 145 \text{ ft.-lbs.}$$

Or, the energy might be determined by computing the algebraic sum of the works done by the external forces acting on the weights, as follows: Let s_a = the distance through which A moves, s_b = the distance through which B moves, and θ = the total angle through which the wheel turns in 5 secs.

Then $$E = 50 \times s_b - 20\,s_a - \frac{10}{12}\theta. \qquad \qquad (6)$$

$$s_a = \frac{1}{2}v_a t = \frac{100\,g}{89} = 36.2 \text{ ft.}$$

$$s_b = \frac{1}{2}v_b t = \frac{50\,g}{89} = 18.1 \text{ ft.}$$

$$\theta = \frac{1}{2}\omega t = \frac{120\,g}{89} = 43.4 \text{ rad.}$$

Substituting these values in equation (6) we obtain

$$E = \frac{400\,g}{89} = 145 \text{ ft.-lbs.}$$

Problem 79.

The weights of 1000 lbs. and 500 lbs. are attached to the ends of a flexible band running over a solid disc, 4 ft. diameter and weighing 1200 lbs., mounted on a shaft A (Fig. 234). Neglecting friction and the weight of the cord and shaft, determine:

(a) The acceleration of each weight and the angular acceleration of the wheel.

(b) The tension in the band between each weight and the wheel.

(c) The kinetic energy of the system after 5 secs., assuming the initial speed of the disc to be 30 revolutions per minute in the direction of the arrow.

(d) The kinetic energy of the system after 5 secs., assuming the initial speed of the disc to be 30 revolutions per minute in a direction opposite to that of the arrow.

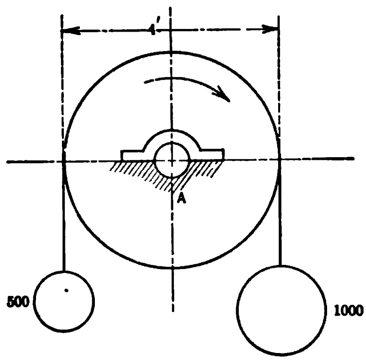

FIG. 234.

Problem 80.

The weight A, of 200 lbs., resting on the inclined plane DE is attached by a flexible cord running over the drum C to the weight B, of 400 lbs., and moves

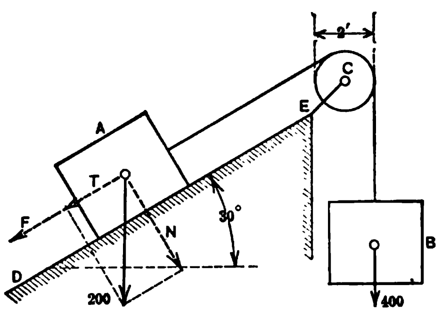

FIG. 235.

under the action of a constant force F parallel to the plane DE (Fig. 235). Assuming that the coefficient of friction between the weight and the plane is constant and equal to 0.25, that the moment of inertia of the drum C is equal to 400 lbs. (ft.)^2, and that the system of weights starts from rest, determine:

(a) The magnitude of the force F necessary to move the weight A *down* the plane 20 ft. in 5 secs.

(b) The magnitude of the force F necessary in order that the weight shall move *up* the plane 20 ft. in 5 secs.

(c) Determine the coefficient of friction between the cord and the drum C, necessary to prevent slipping in each case. (See Art. 191.)

Problem 81.

A solid cylinder A, weighing 200 lbs., is connected by a flexible cord running over the pulley C with a weight B, of 120 lbs. (Fig. 236).

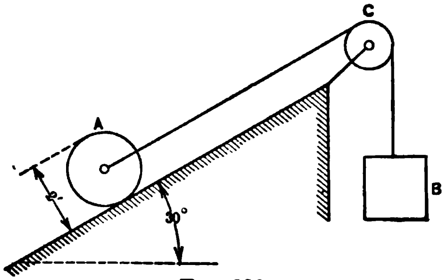

Fig. 236.

Assuming that the weight starts from rest and that the cylinder A rolls without slipping, find the distance through which it will roll in 10 secs. Neglect the weight of the cord and pulley C and rolling friction. Determine the tension in the cord and the energy of the system at the end of 10 secs.

Problem 82.

If the weight B (Problem 81) is 80 lbs., the diameter of the wheel C is 20 ins. and its moment of inertia about its axis 4000 lbs. (in.)^2, determine the distance through which A will roll in 10 secs. Neglect the weight of the cord and rolling friction and assume a constant force of friction at the axis of C, the moment of which is equal to 20 in.-lbs.

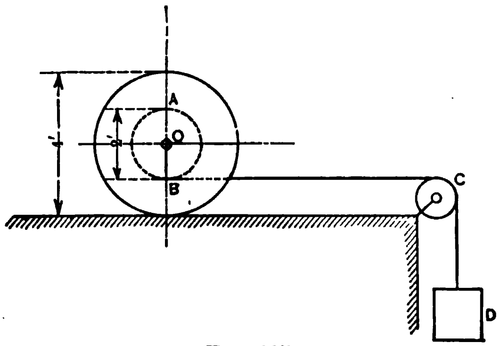

Fig. 237.

Problem 83.

Two circular discs, 6 ins. wide and 4 ft. diameter, are connected by a solid cylinder A, 1 ft. long and 2 ft. diam., as shown. (Fig. 237.) The material in the disc and cylinder weighs 450 lbs. per cu. ft. A weight D of 500 lbs. is attached to a flexible cord which runs over the pulley C and is wound around the cylinder A. If the system starts from rest and the discs roll without slipping how far will the center O travel in 6 secs.? Neglect rolling friction and the weight of the pulley C and the cord. Find the kinetic energy of the system of weights at the end of 6 secs.

§ 5. Impact.

215. Impact, or Collision. — When two bodies collide, a certain amount of deformation takes place, beginning at the instant the bodies come in contact and increasing to a maximum. The deformation may be permanent or, after the collision, the bodies may recover their original forms. Between these two extremes there are numberless cases where there is only a partial recovery of form after the impact. In any case the force exerted between the bodies acts for a very short duration of time and it is practically impossible to determine its magnitude at any instant during that time. The effect of the impact on the momenta and kinetic energies of the bodies may be determined in some cases on the basis of the hypothesis stated in the following article.

While in our work in kinetics thus far we have treated all bodies as rigid, in dealing with impact we shall take into account the elasticity of the colliding bodies in so far as it affects their maximum and final deformation. No attempt will be made, however, to deal with cases where the deformation is great enough to alter the configuration of the mass of either body sufficiently to change its type of motion. On account of the short duration of the contact between the colliding bodies, the force exerted during the contact is frequently called an *impulsive force*, and its effect on the momentum is referred to as an *impulse* (see Art. 148).

216. Types of Impact. — When two bodies collide, unless the contours of the surfaces in contact are identical, they touch at first at a single point only and then, as the deformation takes place, they come in contact over a definite area. According to the hypothesis generally adopted, as soon as the contact begins the bodies begin to be compressed and the compression increases during a certain interval of time, called the "*period of compression*," until it reaches a maximum, at which time the bodies move with the same velocity. After the maximum compression is reached, the bodies in most cases begin to recover their original forms, the recovery continuing until, or even after the contact ceases. The interval of time during which the recovery occurs is called the "*period of restitution*."

A line perpendicular to a plane tangent to the surfaces in contact, at the point through which the resultant pressure between

them acts, is called the *line of impact*. If the contact begins at a single point the line evidently passes through that point.

If the line of impact passes through the centers of gravity of the colliding bodies, the impact is called *central;* if not, it is said to be *eccentric.*

If the line of impact coincides with the direction of the motion, the impact is said to be *direct;* if not, it is called *oblique*. If the recovery of form of the colliding bodies is complete at the end of the contact, the impact is called *elastic;* if the recovery of form is partial, the impact is *imperfectly elastic;* and, if there is no recovery whatever, the impact is *inelastic.*

217. Direct Central Impact. — Let M_1 and M_2 be the masses of two bodies which collide so as to produce direct central impact (Art. 216). Two homogeneous spheres, moving along the line between their centers, furnish a simple illustration under this case. Let their velocities at the beginning of contact be equal to u_1 and u_2, respectively; and let v_1 and v_2 equal their respective velocities at the end of the contact.

In the deduction of the equations in this and the following articles, the velocities u_1, u_2, v_1 and v_2 will all be assumed in the same direction and positive; then if any one of them is in the opposite direction, it will have a negative value.

Let F equal the magnitude of the force exerted between the bodies at any instant during the impact. This force will increase from zero, at the instant the contact begins, to a maximum and then decrease to zero again at the end of the impact.

From the law of action and reaction, it is evident that the change in momentum $\int F\,dt$, produced in each of the two bodies by the force F, must be the same in magnitude and opposite in direction. It is practically impossible in this case to integrate the quantity $\int F\,dt$ but, since each of the bodies has a motion of translation before and after the impact, the value of the integral (Art. 202) may be written

$$M_1 (v_1 - u_1) = - M_2 (v_2 - u_2), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where the change in momentum in the mass M_1 is considered positive.

By transposing equation (1) we obtain

$$M_1 v_1 + M_2 v_2 = M_1 u_1 + M_2 u_2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the same manner, equation (2) would be obtained if v_1 and v_2 were taken to represent the velocities at any instant during the impact. Hence it follows, *that the sum of the momenta of the two colliding bodies remains unchanged during, and after, the impact.*

Equation (2) might have been written as an expression of the second law of motion, since the only forces affecting the motion of the bodies are the mutual action and reaction between them during the impact, which must produce equal and opposite changes in momenta and hence have no effect upon the total momentum of the system.

218. Coefficient of Restitution. — Following the hypothesis already stated (Art. 216), if we let $v =$ the common velocity of the two bodies at the instant of greatest compression and assume it to be positive, the change in velocity of the mass M_1 during the period of compression (Art. 217) will be equal to

$$v - u_1$$

the change being assumed to be positive. Likewise, the change in velocity during the period of restitution will be equal to

$$v_1 - v.$$

The ratio of the second to the first of these two quantities is called the *coefficient of restitution* and will be denoted by the letter e. Hence

$$e = \frac{v_1 - v}{v - u_1}. \qquad (1)$$

Similarly, for the mass M_2

$$e = \frac{v_2 - v}{v - u_2}. \qquad (2)$$

Transposing, we obtain from equation (1)

$$v + ev = v_1 + eu_1,$$

and from equation (2)

$$v + ev = v_2 + eu_2,$$

and hence,

$$v_1 + eu_1 = v_2 + eu_2. \qquad (3)$$

Equation (3), and equation (2) (Art. 217), give the necessary conditions for determining the velocities of the colliding bodies after impact, when the coefficient of restitution is known.

If the bodies are absolutely elastic, $e = 1$; if absolutely inelastic, $e = 0$; and if imperfectly elastic, $e < 1$ and > 0.

By transposing equation (3) we obtain

$$v_1 - v_2 = -e(u_1 - u_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$e = -\frac{v_1 - v_2}{u_1 - u_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which gives the value of e in terms of the differences of the velocities of the bodies before and after the impact.

Hence we may also define the quantity e as *the ratio of the relative velocities of the colliding bodies before and after the impact;* and its value may be determined for bodies of different materials as well as those of the same material.

Newton found, by an elaborate series of experiments with spheres in direct central impact, that their relative velocities before and after impact depended only on the material and followed the law represented by equation (4). The results of his experiments furnished the basis of the hypothesis on which the previous definition of e was based.

219. Kinetic Energy of the Colliding Bodies. — The total kinetic energy of the two bodies in the system before impact will be equal to

$$E_1 = \frac{1}{2} M_1 u_1^2 + \frac{1}{2} M_2 u_2^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and after the impact the energy will be equal to

$$E_2 = \frac{1}{2} M_1 v_1^2 + \frac{1}{2} M_2 v_2^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence the change in the kinetic energy of the system, due to the impact, will be equal to

$$E_1 - E_2 = \frac{1}{2}(M_1 u_1^2 + M_2 u_2^2) - \frac{1}{2}(M_1 v_1^2 + M_2 v_2^2). \quad . \quad (3)$$

The velocities v_1 and v_2 may be eliminated from equation (3) as follows: By squaring equation (2) (Art. 217) we obtain

$$(M_1 v_1 + M_2 v_2)^2 = (M_1 u_1 + M_2 u_2)^2; \quad . \quad . \quad . \quad . \quad (4)$$

and by squaring equation (4) (Art. 218), and multiplying by $M_1 M_2$,

$$M_1 M_2 (v_1 - v_2)^2 = M_1 M_2 e^2 (u_2 - u_1)^2,$$

which may be written in the form

$$M_1 M_2 (v_1 - v_2)^2 + (1 - e^2) M_1 M_2 (u_2 - u_1)^2 = M_1 M_2 (u_2 - u_1)^2. \quad (5)$$

Finally, by adding equations (4) and (5) and reducing, we obtain

$$(M_1 + M_2)(M_1v_1^2 + M_2v_2^2) + (1 - e^2) M_1M_2 (u_2 - u_1)^2$$
$$= (M_1 + M_2)(M_1u_1^2 + M_2u_2^2)$$

and, by transposing and dividing by 2,

$$\frac{1}{2}(M_1u_1^2 + M_2u_2^2) - \frac{1}{2}(M_1v_1^2 + M_2v_2^2)$$
$$= \frac{1}{2}(1 - e^2)\frac{M_1M_2}{M_1 + M_2}(u_2 - u_1)^2 = E_1 - E_2. \quad . \quad (6)$$

When $e = 1$, that is, for perfectly elastic impact,

$$E_1 - E_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and there is no loss in kinetic energy. In all other cases a certain amount of energy is spent in producing the permanent deformation in the colliding bodies, the amount of "lost" energy depending on the value of e. The loss becomes

$$E_1 - E_2 = \frac{1}{2}\frac{M_1M_2}{M_1 + M_2}(u_2 - u_1)^2 \quad . \quad . \quad . \quad . \quad (8)$$

in the case of perfectly inelastic impact.

220. Experimental Determination of the Coefficient of Restitution. — If a small sphere of any substance is allowed to fall vertically upon the horizontal surface of a large mass of the same material, we have a special case of direct central impact which will furnish the data necessary to determine the value of e, the coefficient of restitution for the material. If the mass on which the ball strikes is rigidly supported, its velocity at all times may be assumed to be equal to zero and hence the velocity of both masses at the instant of greatest compression will be equal to zero.

If we let u_1 = the velocity of ball at the beginning and v_1 = its velocity at the end of the impact, and substitute $v = 0$ in equation (1) (Art. 218), we shall have

$$e = -\frac{v_1}{u_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In making the experiment to determine e, the height H, from which the ball falls, and the height h, to which it rises on the rebound, are measured. Then

$$u_1 = \sqrt{2\,gH}$$

and $$v_1 = -\sqrt{2\,gh}.$$

Hence $$e = \sqrt{\frac{h}{H}}. \qquad (2)$$

221. Oblique Central Impact. — (*a*) *When there is no friction.* In this case the resultant force acting between the two bodies during the impact will act along the line of impact, and, if the velocities before and after the impact are resolved into components along the line of impact and perpendicular to it, the latter components will remain unchanged by the collision. The components of the velocity parallel to the line of impact will be affected in the same manner as in the case of direct central impact, and may be determined by means of the formulas deduced for the case of direct central impact (Arts. 217 and 218). It follows that the components of the momenta of the two bodies in the direction perpendicular to the line of impact will remain unchanged by the impact and, that the sum of the components of their momenta in the direction parallel to the line of impact will remain unchanged during and after the impact.

From these conditions, together with that given by equation (3) (Art. 218), the components of the velocities of the two bodies, and hence their resultant velocities after impact, may be determined.

Since the components of the velocities perpendicular to the line of impact remain unchanged, the *loss in kinetic energy* caused by the impact will depend only on the components of the velocities along the line of impact. Hence it will be expressed by the same formulas, and the amount of the loss will be governed by the same conditions as in the case of direct central impact (Art. 219) (see also Art. 174).

(*b*) *When there is friction.* — The effect of friction, at the surfaces in contact when oblique impact occurs, will be to change the components of the velocities of the colliding bodies in the direction perpendicular to the line of impact. If f = the coefficient of friction and F = the normal pressure between the bodies at any instant during the impact, the force of friction at that instant will be equal to fF. This force will act tangent to the surface of contact and it may be resolved into a couple and an equal and parallel force acting through the center of gravity of either body. The effect of the latter component on the momentum of the

mass M_1 (Art. 217) during the impact may be represented by the expression

$$\int fF\,dt = M_1\,(v_{1_t} - u_{1_t}),$$

and similarly, the effect of the component at the center of gravity of the mass M_2 may be represented by the expression

$$-\int fF\,dt = -M_2\,(v_{2_t} - u_{2_t}),$$

where v_{1_t}, v_{2_t}, u_{1_t} and u_{2_t} are the tangential components of the velocities before and after the impact. Hence

$$M_1 v_{1_t} + M_2 v_{2_t} = M_1 u_{1_t} + M_2 u_{2_t}$$

and it follows, therefore, that the sum of the linear momenta of the two bodies in the direction perpendicular to, as well as in the direction parallel to, the line of impact remain unchanged during and after the impact.

The component couples, due to the friction, will produce rotation, depending on the moments of inertia of the two bodies about the axes through their centers of gravity. If any slipping occurs during the impact, a certain amount of kinetic energy will be lost in overcoming the friction. Owing to the uncertainty as to the magnitude of the friction, a more exact analysis of this case is of little value.

222. Direct Eccentric Impact. — *Bodies suspended from fixed axes.* Let W_1 and W_2 equal the weights of two bodies, which collide when rotating about the fixed axes O_1 and O_2, and let the positions of O_1 and O_2 be such that the perpendicular distance of the line of impact from O_1 is equal to L_1 and from O_2 is equal to L_2.

Let I_1 = the moment of inertia of the weight W_1 about the axis O_1 and I_2 = the moment of inertia of the weight W_2 about the axis O_2.

Let μ_1 and μ_2 equal the respective angular velocities of W_1 and W_2 at the beginning of the impact and ω_1 and ω_2 equal the respective angular velocities at the end of the impact.

As in the case of central impact, the forces of action and reaction between the two bodies will be equal at each instant during the impact and will always act along the line of impact. Hence for the weight W_1

$$\int M_1\,dt = L_1 \int F\,dt = \frac{I_1}{g}(\omega_1 - \mu_1) \text{ (Art. 206)}, \quad . \quad . \quad (1)$$

where $M_1 = FL_1$ is the moment at any instant of the force F, due to the impact, about the axis O_1. Similarly, for the weight W_2

$$-\int M_2\,dt = -L_2\int F\,dt = -\frac{I_2}{g}(\omega_2 - \mu_2), \quad . \quad . \quad . \quad (2)$$

where $M_2 = FL_2$ is the moment of the force F about the axis O_2. In equations (1) and (2) the angular velocities are all assumed to be positive and, when the rotation of either body is in the opposite direction to that assumed, its angular velocity is to be treated as a negative quantity. It follows from the law of equality of action and reaction that

$$\frac{I_1}{L_1 g}(\omega_1 - \mu_1) = -\frac{I_2}{L_2 g}(\omega_2 - \mu_2)$$

and by transposing we obtain

$$\frac{I_1\omega_1}{L_1 g} + \frac{I_2\omega_2}{L_2 g} = \frac{I_1\mu_1}{L_1 g} + \frac{I_2\mu_2}{L_2 g}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we denote the linear velocities, before and after the impact, of the points of contact of the two bodies as follows:

$$u_1 = \mu_1 L_1, \qquad u_2 = \mu_2 L_2,$$
$$v_1 = \omega_1 L_1, \qquad v_2 = \omega_2 L_2;$$

equation (3) may be written

$$\frac{I_1 v_1}{L_1^2 g} + \frac{I_2 v_2}{L_2^2 g} = \frac{I_1 u_1}{L_1^2 g} + \frac{I_2 u_2}{L_2^2 g}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

On the basis of the hypothesis, assumed in the case of central impact, we have also the relation deduced in Art. 218,

$$v_1 + eu_1 = v_2 + eu_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The solution of equations (4) and (5) will give the velocities of the two points of impact after the collision when the remaining quantities are known.

223. Loss of Energy. — The total kinetic energy of the two bodies at the beginning of the impact will be equal to

$$E_1 = \frac{\mu_1^2 I_1}{2\,g} + \frac{\mu_2^2 I_2}{2\,g} = \frac{1}{2}\left[\frac{u_1^2 I_1}{L_1^2 g} + \frac{u_2^2 I_2}{L_2^2 g}\right] \quad . \quad . \quad . \quad . \quad (1)$$

and the total kinetic energy of the two bodies at the end of the impact will be equal to

$$E_2 = \frac{\omega_1^2 I_1}{2\,g} + \frac{\omega_2^2 I_2}{2\,g} = \frac{1}{2}\left[\frac{v_1^2 I_1}{L_1^2 g} + \frac{v_2^2 I_2}{L_2^2 g}\right]. \quad . \quad . \quad . \quad . \quad (2)$$

The loss of energy due to the impact will be equal to

$$E_1 - E_2 = \frac{1}{2}\left[\frac{u_1^2 I_1}{L_2^2 g} + \frac{u_2^2 I_2}{L_2^2 g}\right] - \frac{1}{2}\left[\frac{v_1^2 I_1}{L_1^2 g} + \frac{v_2^2 I_2}{L_2^2 g}\right]. \quad . \quad . \quad (3)$$

By the same transformation as that in Art. 219, we may obtain the expression for the loss of energy,

$$E_1 - E_2 = \frac{1}{2}(1 - e^2)\frac{\frac{I_1}{L_1^2 g} \times \frac{I_2}{L_2^2 g}}{\frac{I_1}{L_1^2 g} + \frac{I_2}{L_2^2 g}}(u_2 - u_1)^2$$

$$= \frac{1}{2}(1 - e^2)\frac{I_1 I_2}{g\,(I_1 L_2^2 + I_2 L_1^2)}(u_2 - u_1)^2, \quad . \quad . \quad . \quad (4)$$

which shows that, when the impact is perfectly elastic, no loss in kinetic energy results, and that in all other cases the loss depends on the value of e.

224. Oblique Eccentric Impact. — As in the case of oblique central impact, if the friction between the colliding bodies is neglected, and the velocities of the points of contact are resolved into components perpendicular and parallel to the line of impact, the former components will remain unchanged by the impact while the latter can be computed by means of the formulas in Art. 222.

225. Central and Eccentric Impact. — In certain cases, the impact of one of the colliding bodies may be central and of the other, eccentric. The formulas for such cases may be deduced by the same method of reasoning as in the preceding articles. Thus, in the case where a body, moving in a straight line, strikes another, suspended from a fixed axis, in such a manner that the impact on the first body is direct central and on the second direct eccentric, the formulas, in the notation adopted in the preceding articles, will be the following:

$$M_1 v_1 + \frac{J_2 v_2}{L_2^2 g} = M_1 u_1 + \frac{I_2 u_2}{L_2^2 g}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$v_1 + e u_1 = v_2 + e u_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$E_1 - E_2 = \frac{1}{2}\left[M_1 u_1^2 + \frac{I_2 u_2^2}{L_2^2 g}\right] - \frac{1}{2}\left[M_1 v_1^2 + \frac{I_2 v_2^2}{L_2^2 g}\right]. \quad . \quad . \quad (3)$$

An illustration under this case is the ballistic pendulum (Art. 229).

226. Examples of Direct Central Impact. — Except in the case of a few laboratory experiments, it will be found that in nearly all practical examples of impact the determination of the velocities of the colliding bodies after impact would be of little value, even if the uncertainty regarding the value of e, which usually exists in such cases, did not make the determination impossible.

In such cases, however, the kinetic energy of each of the impinging bodies at the beginning of the impact can be determined with a fair degree of accuracy and, from a general knowledge of the material and the masses of the two bodies, the manner in which the energy will be expended can be predicted.

Thus: in the case of the steam hammer, the piston, piston rod and head comprise the mass M_1, of one of the colliding bodies, while the mass M_2 may be considered to be composed of the forging, the anvil and its foundations. The impact may be assumed to be direct and central and, on account of the plasticity of the hot metal, the value of e will be nearly zero.

Following the notation in the preceding articles: the value of u_1 may be computed when the steam pressure and the height of the drop are known; the value of $u_2 = 0$, practically (actually, the Earth and the hammer approach each other with equal momenta); and, on the assumption that $e = 0$, $v_1 = v_2 = 0$.

The "*energy of the blow*" is the kinetic energy of the mass M_1 at the beginning of the impact and, on account of the small value of e, this is largely expended in compressing the forging.

The "*force of the blow*" is the force F, exerted between the hammer and the forging during the impact. It is a varying quantity and cannot be accurately determined. If $e = 0$ and s = the distance the forging is compressed, the average value of F may be roughly determined from the equation

$$Fs = E_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The expression "force of the blow" is sometimes used erroneously when the "energy of the blow" is meant.

Another example of direct central impact is that of the pile driver. As in the case of the steam hammer, an accurate determination of the momentum of the hammer and the pile after impact cannot be made.

The "energy of the blow" E_1 can be easily figured from the drop of the hammer. A large part of this energy will be expended in useful work in driving the pile, but some of it will be lost in compressing the top of the pile and a small amount is usually lost in the rebound of the hammer.

As in the preceding case the average "force of the blow" may be roughly determined by equation (1) if we assume $e = 0$ and $s =$ the distance the pile settles under the blow.

227. Recoil of a Gun. — The explosion of powder in a gun produces impulsive forces, acting on both the shot and the gun which, if the mass of the powder and gases are neglected, are always equal and opposite. The case is similar, therefore, to that of the direct central impact of two colliding bodies. If we let $M_1 =$ the mass of the shot and $M_2 =$ the mass of the gun, we obtain from equation (2) (Art. 217)

$$M_1v_1 + M_2v_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and hence

$$v_1 = -\frac{M_2}{M_1}v_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or,

$$v_2 = -\frac{M_1}{M_2}v_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The total kinetic energy due to the explosion of the powder will be equal to

$$\frac{1}{2}M_1v_1^2 + \frac{1}{2}M_2v_2^2 = \frac{M_2 + M_1}{M_1} \times \frac{1}{2}M_2v_2^2 = \frac{M_1 + M_2}{M_2} \times \frac{1}{2}M_1v_1^2. \quad . \quad (4)$$

Therefore, the kinetic energy of the shot will be equal to $\frac{M_2}{M_1 + M_2}$ of the total energy of the explosion and the energy of the recoil, to $\frac{M_1}{M_1 + M_2}$ of the total energy.

228. Impact of a Stream of Particles on a Flat Plate. — If a continuous stream of very small particles moving in parallel paths with equal velocities is allowed to strike against a flat plate, the resultant pressure required to prevent a movement of the plate under the series of impacts may be determined in the following manner:

Let m_1, m_2, m_3, etc. equal the masses of the particles and u, their common velocity before impact and assume that the impact of each particle is inelastic, that is, $e = 0$. Two cases will be

considered: (*a*) when the impact is direct; (*b*) when the impact is oblique.

(*a*) *Direct impact.* In this case the velocities of all the particles after impact will be equal to zero and, if n particles strike the plate in the time t, the sum of the impulses during that time will be equal to

$$m_1u + m_2u + \cdots + m_nu = (m_1 + m_2 + \cdots + m_n)\,u = u\Sigma m$$
$$= \int F_1dt + \int F_2dt + \cdots + \int F_ndt, \quad . \quad . \quad . \quad . \quad (1)$$

where F_1, F_2, etc., are the forces of impact of the separate particles. If we let R equal the average reaction exerted by the plate, equation (1) may be written

$$u\Sigma m = \Sigma \int F\,dt = Rt \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and, if $t = 1$ second,

$$R = u\Sigma m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where Σm = the total mass of the particles striking the plate in one second.

(*b*) *Oblique impact.* If we let θ = the angle between the paths of the particles and the normals at their points of impact, the normal components of their velocities will be equal to $u \cos \theta$ and hence, by substituting in equations (2) and (3), we obtain

$$Rt = u \cos \theta \Sigma m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and, if $t=1$,

$$R = u \cos \theta \Sigma m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where R = the normal reaction of the plate. If there is no friction, the tangential components of the velocities of the particles will remain unchanged and there will be no tangential reaction along the surface of the plate. If there is friction, this component will depend on the obliquity of the impact and the coefficient of friction (Art. 221).

229. Ballistic Pendulum. — The ballistic pendulum is a device which has been used for determining the velocity of a projectile. It consists of a large mass M_2 which is suspended from a horizontal axis O (Fig. 238). In order to render the impact as inelastic as possible, a cavity on the side on which the projectile strikes is filled with wood or other soft material, in order that the projectile may remain embedded in the mass and oscillate with it after the impact. The device furnishes an illustration of a combined

direct central and eccentric impact (Art. 225) which is inelastic. Thus: let M_1 = the mass of the projectile and u_1 = its velocity at the beginning of the impact; let I_2 = the moment of inertia of the mass M_2, in terms of its weight, about the axis O, let L_2 = the perpendicular distance of the line of impact from O and let $v = v_1 = v_2$ the linear velocity of the point of impact. During the impact the motion of M_2 will be so slight that the velocity v may be considered to be horizontal, without introducing any appreciable error. After the impact, the pendulum will oscillate about O and its center of oscillation (Art. 210) will rise through a height h which can be easily computed from the measurement of the angle through which the pendulum swings from the vertical.

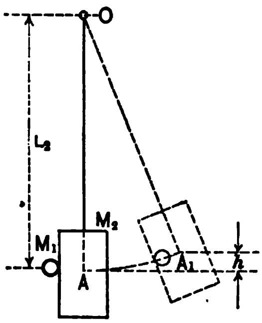

FIG. 238.

To insure accuracy, the line of impact should pass through the center of oscillation. Then, when h is known, v may be determined from the equation for the velocity of the equivalent simple pendulum,

$$v = \sqrt{2\,gh} \text{ (Art. 155).}$$

By substituting in equation (1) (Art. 223) we have

$$M_1 v + \frac{I_2 v}{L_2^2 g} = M_1 u_1, \quad \ldots \ldots \quad (1)$$

from which we obtain

$$u_1 = \left(1 + \frac{I_2}{M_1 L_2^2 g}\right)\sqrt{2\,gh}. \quad \ldots \ldots \quad (2)$$

If the line of impact does not pass through the center of oscillation and the connection between the mass M_2 and the axis O is rigid enough to prevent any other displacement, than the one indicated (Fig. 238), the angular velocity of M_2 at the end of the impact will be equal to

$$\omega = \frac{v_c}{l} = \frac{\sqrt{2\,gh}}{l}, \quad \ldots \ldots \quad (3)$$

where v_c = the velocity of the center of oscillation of the pendulum after the impact and l = its distance from O. Then by substituting the value of v, obtained from the equation

$$\omega = \frac{v}{L_2} = \frac{\sqrt{2\,gh}}{l},$$

equation (1) takes the form

$$\left[M_1L_2 + \frac{I_2}{L_2g}\right]\frac{\sqrt{2\,gh}}{l} = M_1u_1 \quad . \quad . \quad . \quad . \quad (4)$$

and equation (2) the form

$$u_1 = \left[L_2 + \frac{I_2}{M_1L_2g}\right]\frac{\sqrt{2\,gh}}{l}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

230. Problems.—Impact.

Problem 1.

Two weights of 20 lbs. and 40 lbs., moving along the same line in opposite directions with velocities of 10 ft. per second, strike in central impact. If the coefficient of restitution is 0.6, find the velocities after impact; find the loss of kinetic energy during impact.

Problem 2.

Two perfectly inelastic bodies weighing 40 lbs. and 60 lbs., moving in the same direction with velocities of 16 ft. per second and 26 ft. per second, respectively, impinge on each other in central impact; find the loss of kinetic energy due to the impact.

Problem 3.

A freight car, weighing 20 tons, and travelling at a speed of 20 miles per hour overtakes and runs into another car, weighing 15 tons, travelling on the same track in the same direction at a speed of 15 miles per hour. Find the velocity of each car after the impact, assuming $e = 0.2$.

Problem 4.

10 blows of a ram of 2000 lbs., falling from a height of 4 ft., sink a pile, weighing 400 lbs., 4 ins. If the permanent load is taken as one-fifth of the average resistance, what permanent load can the pile bear?

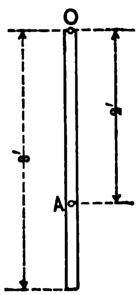

FIG. 239.

Problem 5.

A steam hammer of 2 tons is used in forging. It has a fall of 4 ft. If the piston is 12 ins. diameter and the mean effective steam pressure during the stroke is 80 lbs. per sq. in., find the energy of the blow delivered by the hammer, neglecting piston friction.

Problem 6.

A straight rod of uniform section, weighing 200 lbs. and suspended at O (Fig. 239) is struck at A by a ball weighing 8 lbs., moving in a horizontal direction with a velocity of 10 ft. per second. Find the angle through which the rod will swing. Assume $e = 0.5$.

Problem 7.

A sphere weighing 10 lbs., suspended by a cord from B (Fig. 240), starts from rest and swings through 45° to a vertical position, when it strikes the sphere weighing 20 lbs., suspended at rest from A. Find the angle through which each weight will swing after impact. Assume $e = 0.8$.

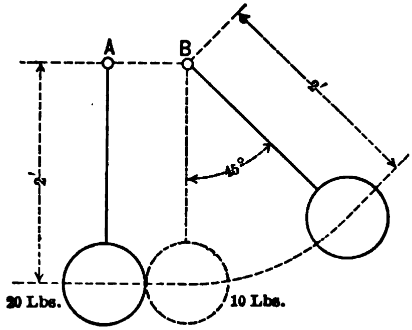

FIG. 240.

Problem 8.

Two spheres, weighing 20 lbs. and 10 lbs., travel towards each other, in straight horizontal lines which intersect at an angle of 120°, with velocites of 15 ft. per second and 20 ft. per second, respectively. Find the magnitudes and directions of their respective velocities after impact, assuming $e = 0.8$ and neglecting friction.

Find the loss of energy due to the impact.

Problem 9.

A solid sphere weighing 20 lbs. has imparted to it an initial velocity of 20 ft. per sec. in the horizontal direction and falls as a freely moving projectile, striking a smooth horizontal surface 30 ft. below the level of its starting point. How far will it travel in the horizontal direction before striking? If $e = 0.5$, how far will it travel in the horizontal direction before striking the surface again after the rebound? Neglect all frictional resistances.

Problem 10.

A straight homogeneous rod of uniform section 4 ft. long, weighing 20 lbs., is mounted on a pivot at O so as to swing in a horizontal plane (Fig. 241). If the rod is struck at the point A by a ball weighing 5 lbs., traveling with a velocity of 20 ft. per second in a horizontal path, making an angle of 60° with the rod, find the angular velocity of the rod after impact. Assume $e = 0.6$. If the moment of friction on the pivot is 5 in. lbs., how long will the rod spin before stopping?

Problem 11.

A ball weighing 10 lbs. falls from a height of 6 ft. and strikes on a horizontal surface. If $e = 0.6$, how high will the ball rise on the second rebound? Find the loss of energy at the end of the third impact.

Problem 12.

A shot weighing 600 lbs. is fired with a velocity of 1600 ft. per sec. from a gun weighing 40 tons. Find the velocity of the recoil, neglecting the weight of the powder. If the recoil is resisted by a constant pressure of 20,000 lbs. through what space will the gun move?

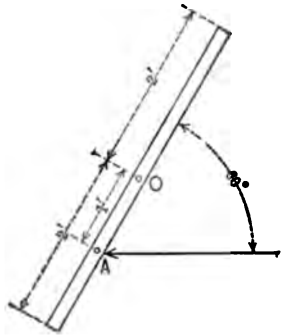

FIG. 241.

Problem 13.

If the projectile, in Problem 12, were fired under the same conditions from a gun weighing 30 tons, through what space would the gun move on the recoil?

Problem 14.

If 500 shots per minute are fired from a machine gun, in the horizontal direction, and strike against a vertical target with a velocity of 2000 ft. per sec., find the horizontal pressure required to support the target. Assume that the weight of each projectile is 1 oz. and that its velocity after impact is equal to zero.

Problem 15.

A straight rod, of uniform section and material and weighing 24 lbs. is supported on a horizontal axis through its center, about which it is free to turn in a vertical plane. Another similar rod 2 ft. long, weighing 10 lbs., is suspended from a parallel axis through one end: and the position of the axis is such that, if the rod is allowed to swing through 90 degrees from a horizontal position, its lower end will strike the top of the first rod a blow in the horizontal direction. If $e = 0.5$ find the angular momentum of each rod after impact.

APPENDIX.

205a. Rotation of Rigid Bodies About Fixed Axes. — In Art. 205 is given the general discussion of this proposition and from the equations there deduced, the formulas for the resultant of any system of forces, acting on any rigid body and producing rotation about a fixed axis, may be found.

These expressions are very much simplified in the case of the slender straight rod and the thin flat plate and, as these two cases may be considered as fundamental in character, independent solutions will now be given.

(*a*) *Slender Straight Rod.* — Let CD be a slender straight rod which has imparted to it a motion of rotation about a fixed axis perpendicular to its length through O (Fig. 208a).

Assume the rectangular coördinate axes OX, OY and OZ with OX in the direction of the length of the rod and OZ coinciding with the axis of rotation. Let A be the center of gravity of the rod, M = its mass and W = its weight. Let ω = the angular velocity and α = the angular acceleration at any instant.

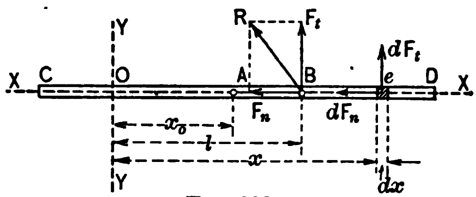

FIG. 208a.

The resultant of the external forces acting on the rod at any instant can be found as follows:

Let dM = the mass of a particle of length dx at any distance x from the axis of rotation through O. Let v = the linear velocity of the particle and a = its linear acceleration.

The direction of v will be perpendicular to OX and a may be resolved into a tangential component a_t, perpendicular to OX, and a normal component a_n, along OX.

If we resolve the resultant force acting on the particle into a tangential component dF_t and a normal component dF_n (Art. 150), we shall have

$$dF_t = a_t\,dM = \alpha x\,dM \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$dF_n = \frac{v^2}{x}\,dM = \omega^2 x\,dM \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By integrating equation (1), we obtain the expression for the resultant of the tangential forces acting on all the particles of the rod

$$\Sigma Y = \int dF_t = \alpha \int x\, dM; \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and by integrating equation (2), the expression for the resultant of the normal components acting on all the particles of the rod

$$\Sigma X = \int dF_n = \omega^2 \int x\, dM. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If we let $\int dF_t = F_t$, equation (3) reduces to

$$F_t = \alpha \int x\, dM = \alpha x_0 M \text{ (Art. 92)}, \quad . \quad . \quad . \quad . \quad (5)$$

and if we let $\int dF_n = F_n$, equation (4) reduces to

$$F_n = \omega^2 \int x\, dM = \omega^2 x_0 M. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

But $\alpha x_0 = a_0$, the tangential component of the acceleration of the center of gravity of the rod and $\omega x_0 = v_0$, the linear velocity of the center of gravity; hence

$$F_t = M\alpha x_0 = Ma_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and

$$F_n = M\omega^2 x_0 = M\frac{v_0^2}{x_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Therefore the resultant of the force system required to impart the motion will be equal to

$$R = \sqrt{(F_t)^2 + (F_n)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and will evidently be of *the same magnitude as if the entire mass of the rod were concentrated at its center of gravity.*

It is also evident that the line of action of the component force F_n will be along the center line of the rod in a direction perpendicular to the axis of rotation.

Finally, we must determine the line of action of the component F_t. The moment of the force dF_t, acting on a particle dM, about the axis OZ will be equal to

$$x\, dF_t = a_t x\, dM = \alpha x^2\, dM$$

and the sum of the moments of the forces acting on all the particles in the rod will be equal to

$$\Sigma M_z = \int x\, dF_t = \alpha \int x^2\, dM = \alpha I_m. \quad . \quad . \quad . \quad . \quad (10)$$

If we let $\Sigma M_s = M_0$ and substitute for I_m its value $\frac{I}{g}$ in terms of the moment of inertia in units of weight we obtain

$$M_0 = \frac{\alpha I}{g}, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

where M_0 is the resultant moment about the axis of rotation of the system of forces producing the motion and I is the moment of inertia about that axis.

If we let l = the distance from O to the point B through which the resultant force passes, we shall obtain

$$l = \frac{M_0}{F_t} = \frac{\frac{\alpha I}{g}}{\alpha x_0 M} = \frac{I}{x_0 W} \text{ (Art. 76)}. \quad . \; . \; . \; . \quad (12)$$

The point B is called the center of percussion of the rod with respect to the axis of rotation through O.

When the axis of rotation passes through the center of gravity, $x_0 = 0$ and the resultant of the tangential components

$$F_t = M\alpha x_0 = 0 \quad . \; . \; . \; . \; . \; . \quad (13)$$

and of the normal components

$$F_n = M\omega^2 x_0 = 0 \quad . \; . \; . \; . \; . \; . \quad (14)$$

and hence the resultant of the system of forces will be a couple whose moment (equation 11) will be equal to

$$M_0 = \frac{\alpha I_0}{g}, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (15)$$

where I_0 = the moment of inertia of the rod about an axis through its center of gravity.

When the rotation is uniform and the axis passes through the center of gravity, the force system must be balanced and $M_0 = 0$.

When the motion is uniform and O and A do not coincide, the resultant of the system will be equal to the component F_n (equation 8).

For the expressions for the angular momentum, kinetic energy and centripetal force of the rod see Arts. 206, 207 and 208.

(*b*) *Thin Flat Plate.* — Let *abc*, Fig. 208b, be a thin flat plate of any shape and dimensions which has imparted to it a motion of rotation about a fixed axis through O perpendicular to the plane of the plate.

Assume the rectangular coördinate axes OX, OY and OZ with OZ coinciding with the axis of rotation and OX passing through A, the center of gravity of the plate. Let W = the weight of the plate, M = the mass, I_m the moment of inertia in mass units of the plate about the axis OZ and I = the moment of inertia about OZ in units of weight. Let ω = the angular velocity and α = the angular acceleration at any instant and let $x_0 = OA$ the perpendicular distance from the axis of rotation to the center of gravity. In the case of a very thin plate we may consider all the particles of which it is composed to be located in a single plane and since the motion is in the plane of the plate, the resultant force acting on each particle will act in that plane.

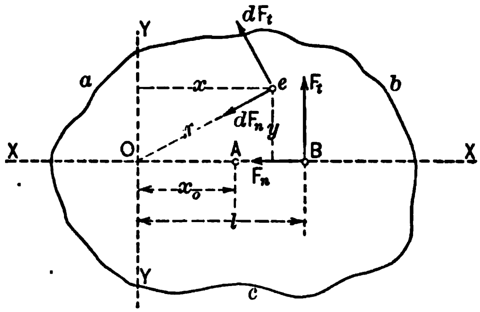

FIG. 208b.

The resultant of the system of external forces acting on the plate at any instant may be found as follows:

Let dM = the mass of any particle e, whose coördinates are (x, y) and whose perpendicular distance from $OZ = r$. The resultant of the force system acting on e may be resolved into a tangential component dF_t and a normal component dF_n, where

$$dF_t = \alpha r\, dM \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and

$$dF_n = \omega^2 r\, dM \text{ (Art. 150)}. \quad . \quad . \quad . \quad (17)$$

To obtain the resultant of the force system acting on the entire plate we must resolve the components dF_t and dF_n acting on each particle into components parallel to OX and OY and determine the vector sum of each set of components.

Resolving dF_t in this manner, we obtain for the component parallel to OX

$$\frac{y}{r} dF_t = \frac{y}{r} \alpha r\, dM = \alpha y\, dM,$$

and for the component parallel to OY

$$\frac{x}{r}\,dF_t = \frac{x}{r}\,\alpha r\,dM = \alpha x\,dM.$$

Resolving the tangential component acting on every particle in the same manner and adding together, we obtain for the vector sum of the components parallel to OX

$$\Sigma X_t = \int \frac{y}{r}\,dF_t = \alpha \int y\,dM = \alpha y_0 M \text{ (Art. 92)}, \quad . \quad (18)$$

and for the vector sum of the components parallel to OY

$$\Sigma Y_t = \int \frac{x}{r}\,dF_t = \alpha \int x\,dM = \alpha x_0 M. \quad . \quad . \quad . \quad (19)$$

Since OX passes through the center of gravity, $y_0 = 0$ and hence

$$\Sigma X_t = 0.$$

If we let F_t = the resultant of the tangential components acting on all the particles of the plate, we shall have

$$F_t = \Sigma Y_t = \alpha x_0 M = M a_0, \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where a_0 = the tangential component of the acceleration of the center of gravity.

Resolving in a like manner the normal component dF_n (equation 17) acting on every particle and adding together, we obtain for the vector sum of the components parallel to OX

$$\Sigma X_n = \int \frac{x}{r}\,dF_n = \omega^2 \int x\,dM = \omega^2 x_0 M, \quad . \quad . \quad (21)$$

and for the vector sum of the components parallel to OY

$$\Sigma Y_n = \int \frac{y}{r}\,dF_n = \omega^2 \int y\,dM = \omega^2 y_0 M. \quad . \quad . \quad (22)$$

Since $y_0 = 0$ $\qquad \Sigma Y_n = 0.$

If we let F_n = the resultant of the normal components acting on all the particles we shall have

$$F_n = \Sigma X_n = \omega^2 x_0 M. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

It is evident from the preceding that the resultant of the entire force system acting on the plate will be a force in the Z plane whose magnitude is equal to

$$R = \sqrt{(F_t)^2 + (F_n)^2} = x_0 M \sqrt{\omega^4 + \alpha^2}. \quad . \quad . \quad (24)$$

To determine the line of action of the resultant we may proceed as follows:

Since the components of the force acting on any particle e are in the Z plane, the moments of each of these components about OX and OY will be zero.

Since the normal component dF_n acting on any particle passes through the axis OZ its moment about OZ will equal zero. The moment of the tangential component dF_t about OZ will be equal to

$$r\,dF_t = \alpha r^2\,dM.$$

Hence the line of action of the resultant F_n of the normal components acting on all the particles in the plate will coincide with the axis OX through the center of gravity of the plate.

The moment of the resultant F_t of the tangential components acting on all the particles about the axis OZ will be equal to

$$\Sigma M_z = \int r\,dF_t = \alpha\int r^2\,dM = \alpha I_m = \frac{\alpha I}{g}. \quad . \quad . \quad . \qquad (25)$$

If we let $M_0 = \Sigma M_z$, $l =$ the perpendicular distance from OZ to the line of action of F_t, we have

$$M_0 = F_t l = \frac{\alpha I}{g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

the expression for the resultant moment about the axis of rotation of the entire system of forces acting on the plate.

Substituting the value of F_t in equation (25) and solving for l we obtain

$$l = \frac{\alpha I_m}{\alpha x_0 M} = \frac{I_m}{x_0 M} = \frac{I}{x_0 W}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

It follows that the resultant F_t is a force parallel to OY intersecting the axis OX at a point B at a distance l from the axis of rotation. The point B is called the center of percussion of the plate with respect to the axis of rotation OZ.

Therefore the resultant of the entire force system acting on the plate is made up of a tangential and a normal component acting through its center of percussion each of which has the same magnitude as if the entire mass were concentrated at the center of gravity of the plate.

An exception occurs when the axis of rotation passes through the center of gravity. In this case $x_0 = 0$ and hence

$$F_t = \alpha x_0 M = 0,$$

and

$$F_n = \omega^2 x_0 M = 0$$

and the resultant of the force system acting on the plate becomes a couple of magnitude

$$M_0 = \frac{\alpha I_0}{g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where I_0 = the moment of inertia in units of weight about the axis through the center of gravity.

It is evident that when the rotation is uniform and the axis passes through the center of gravity the forces acting on the plate form a balanced system and, when the axis does not pass through the center of gravity and the motion is uniform, the resultant of the system is the normal force F_n, where

$$F_n = \omega^2 x_0 M.$$

As in the case of the straight rod, the expression for angular momentum, kinetic energy and centripetal force will be found in Arts. 206, 207 and 208.

Since no restrictions in regard to the shape of the flat plate were imposed in the preceding analysis it is evident that it will apply in the case of a thin disc or thin ring of any shape rotating about an axis perpendicular to its plane.

(c) *Any body symmetrical with respect to its plane of rotation.* — Since any body which is symmetrical with respect to a plane of rotation through its center of gravity may be considered to be made up of a series of thin flat discs symmetrically placed with respect to the plane of rotation, the resultant of the force system acting on the entire body will be the same as if its entire mass were concentrated in the plane of rotation. Many of the problems of rotation of rigid bodies, with which the engineer has to deal, are included in this case and may be solved by using the preceding formulas.

INDEX